

Simplicial branching random walks and their applications

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Abstract

We define a new stochastic process on general simplicial complexes which allows to study their spectral and homological properties. Some results for random walks on graphs are shown to hold in this general setting. As an application, the process is used to calculate the spectral measure of high-dimensional analogues of regular trees and to construct solutions to the high-dimensional Dirichlet problem for forms.

1 Introduction

The topic of “random walks on graphs” is a classical and fundamental subject. With a history of more than a century and a variety of applications to physics, computer science, chemistry and many other fields, random walks are among the most valuable stochastic models. In addition, their connections with many areas of research within mathematics such as probability, geometry, graph theory, harmonic analysis, group theory, etc, make random walks a valuable tool when investigating their interplay. Accordingly, the literature is very vast and we refer the reader to the following books [Spi76, DS84, Lov96, Woe00, LL10] as well as the references therein for background on the subject.

Simplicial complexes are combinatorial and topological extensions of graphs and it is thus natural to ask whether one can generalize random walk models to the world of high-dimensional simplicial complexes.

A first construction of such a stochastic process was suggested in [PR12] by Parzanchevski and the author. The process, which is called the $(d - 1)$ -random walk, reflects in its asymptotic behavior spectral properties of the upper Laplacian (originating in the work of Eckmann [Eck45]) as well as homological properties of the complex. In a subsequent work [MS13], Mukherjee and Steenbergen constructed a similar model for random walks on simplicial complexes, which is connected to the lower Laplacian and in particular allows to study the top homology of the complex.

The connection of both models to the spectral and homological properties of simplicial complexes is done via the study of an associated “process”, called the expectation process, which takes the role played by the heat kernel of a random walk in the graph case. This generalization differs from “classical” heat kernels in two regards: first, due to the fact that high-dimensional simplexes have two possible orientations, it is defined as the difference of two probabilities. Secondly, in order to extract information from this difference of probabilities, which always converges to zero, a suitable normalization is required.

In this paper, we present a new stochastic process, called *simplicial branching random walk*, which is connected to spectral and homological properties of simplicial complexes in a similar way as the other processes. However, unlike in the previous models, the fact that one needs to look at the difference of two quantities can already be observed at the level of the process *itself* and not merely in the context of an associated process. Moreover in this new process, there is no need for normalization. Hence, we can

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work with it *directly*, and are able to gain new insights regarding the nature of this process as well as its connections to spectral and homological properties of the complex.

In recent years, there has been considerable interest in *high-dimensional expanders*, namely, analogues of expander graphs in the context of general simplicial complexes. As in the graph case, stochastic processes like the simplicial branching random walk and the $(d - 1)$ -random walks constructed in [PR12, MS13] are closely related to one such notion of expansion, namely, spectral expansion. Other notions of expansion include: combinatorial expansion [PRT12, Par13, GS14], geometric and topological expansion [Gro10, FGL⁺12, MW14], \mathbb{F}_2 -coboundary expansion [DK12, SKM14] and Ramanujan complexes [CSŽ03, Li04, LSV05, GP14, EGL14, KKL14]. There is also a great interest in the behavior of random complexes. The standard model for such complexes is the Linial-Meshulam model, defined in [LM06], which has been extensively studied, see [MW09, Koz10, Wag11, HKP12, HJ13, HKP13, LP14, GU14]; see also [LM13] for related results on a different model.

1.1 The model

Let us start with an informal description of the the model. A more precise definition is postponed to Section 3 after all required notation and terminology are introduced in Section 2. Let X be a d -dimensional complex. The *simplicial branching random walk* (SBRW for short) $(N_n)_{n \geq 0}$ is a particle process on the set of oriented $(d - 1)$ -simplexes of X , denoted by X_{\pm}^{d-1} , where, for an oriented $(d - 1)$ -simplex σ and $n \geq 0$, $N_n(\sigma)$ stands for the number of particles in σ at time n . The process $(N_n)_{n \geq 0}$ is a time-homogeneous Markov chain on $\mathbb{N}^{X_{\pm}^{d-1}}$, with transition kernel which is described by the following law: Given a configuration of particles, each of the particles (simultaneously and independently) chooses one of the d -simplexes containing the $(d - 1)$ -simplex of its current position uniformly at random and splits into d new particles that are now located on the other d faces of the chosen d -simplex (with an appropriate choice of orientation, see Section 3).

For example, if X is a triangle complex, the SBRW is a particle process on oriented edges. If a particle is positioned on the oriented edge $[u, v]$ and the chosen triangle containing it is $\{u, v, w\}$ then the particle splits into two new particles on $[u, w]$ and $[w, v]$ (the orientation is chosen so that the original oriented edge and the new oriented edges have the same origin or the same terminus).

Given $0 \leq p \leq 1$, one can also discuss the p -lazy version of the SBRW in which every particle stays put with probability p and with probability $(1 - p)$ acts according to the law described above. An illustration of one step of the process for a triangle complex can be found in Figure 1.1.

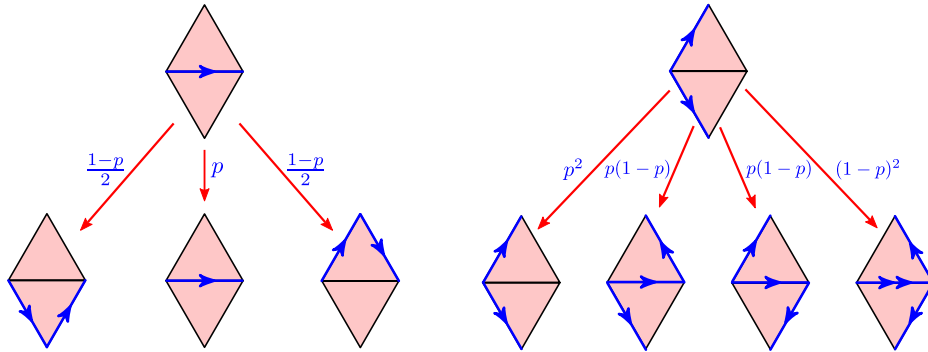


Figure 1.1: One step of the simplicial branching random walk for two configurations. On the left: The particle starting at the center stays put with probability p and with probability $1 - p$ chooses one of the triangles containing it uniformly at random and splits into two particles on the two other (neighboring) edges of this triangle. On the right: each of the particles stays put with probability p or splits into two particles on the unique triangle containing its current edge with probability $1 - p$.

We now introduce the effective version of the process called *effective simplicial branching random walk* (ESBRW for short) by

$$D_n(\sigma) = N_n(\sigma) - N_n(\bar{\sigma}),$$

where for an oriented $(d-1)$ -simplex σ , $\bar{\sigma}$ is the same $(d-1)$ -simplex with the opposite orientation. Finally the heat kernel is defined as

$$\mathcal{E}_n(\sigma, \sigma') = E^\sigma[D_n(\sigma')],$$

where E^σ denotes the expectation when starting with a unique particle on the oriented $(d-1)$ -simplex σ .

1.2 Main results

We now give a description of the main results. For the sake of clarity we only give informal statements for some of the results and refer the reader to later sections for the precise statements.

The first result deals with the connection between the asymptotic behavior of the heat kernel and the existence of non-trivial homology in finite complexes. It is shown that a similar relation to the one proved in [PR12] for the $(d-1)$ -walk holds for the ESBRW (see Theorem 3.1 for the precise statement).

Theorem. *Let X be a **finite** d -complex, $(D_n)_{n \geq 0}$ the p -lazy ESBRW on X and $(\mathcal{E}_n)_{n \geq 0}$ its heat kernel. If $p > \frac{d-1}{d+1}$, then*

- (1) *The limit $\mathcal{E}_\infty = \lim_{n \rightarrow \infty} \mathcal{E}_n$ always exists.*
- (2) *One can read off from the family $\{\mathcal{E}_\infty(\sigma, \cdot)\}_{\sigma \in X_\pm^{d-1}}$ the dimension of the $(d-1)$ -homology, and in particular, whether the homology is trivial.*
- (3) *If furthermore $p \geq \frac{d}{d+1}$, then the rate of convergence of \mathcal{E}_n is exponential with a constant that depends on a high-dimensional analogue of the spectral gap.*

Our next result concerns a generalization of the following important identity for random walks on graphs (see also Theorem 3.6).

Theorem. *Let P^v (E^v) be the law (expectation) of a random walk on a graph G . The identity*

$$E^v \left[\begin{array}{c} \text{number of visits to } v \\ \text{by the random walk} \end{array} \right] = \frac{1}{1 - P^v \left(\begin{array}{c} \text{The random walk} \\ \text{returns to } v \end{array} \right)} \quad (1.1)$$

has a high-dimensional analogue for the effective simplicial branching random walk.

Next, we discuss some applications of ESBRW to the study of simplicial complexes. The d -dimensional counterpart of the k -regular tree, called k -regular arboreal d -complex, was defined in [PR12]. It is obtained by attaching to a $(d-1)$ -simplex k new d -simplexes and then adding recursively to every new $(d-1)$ -simplex $(k-1)$ new d -simplexes (see also Definition 4.1). By generalizing ideas of Kesten [Kes59] to ESBRW, we are able to find the spectral measure of the “transition” operator \mathcal{A}_0 (see Lemma 3.2 for the definition).

Theorem 1.1. *The spectral measure $\mu_{d,k}$ of $\mathcal{A}_0 = I - \Delta^+$ for the k -regular arboreal d -complex is given by*

$$\mu_{d,k}(A) = \begin{cases} \int_A \rho_{d,k}(x) dx + \frac{d+1-k}{d+1} \chi_{1 \in A} & , \quad k < d+1 \\ \int_A \rho_{d,k}(x) dx & , \quad k \geq d+1 \end{cases},$$

where χ is the indicator function,

$$\rho_{d,k}(x) = \frac{\sqrt{4(k-1)d - (kx + (d-1))^2}}{2\pi(d+x)(1-x)} \chi_{x \in I_{d,k}}$$

and

$$I_{d,k} = \left[\frac{1-d-2\sqrt{(k-1)d}}{k}, \frac{1-d+2\sqrt{(k-1)d}}{k} \right].$$

In particular, this gives a new proof of the fact that the spectrum of \mathcal{A}_0 is $I_{d,k}$ for $k \geq d+1$ and $I_{d,k} \cup \{1\}$ when $k < d+1$, which is the content of [PR12, Theorem 3.3].

As a corollary of Theorem 1.1 we obtain the following transience/recurrence classification for regular arboreal complexes:

Corollary 1.2. *The effective simplicial branching random walk on T_k^d is recurrent, i.e., $\sum_{n=0}^{\infty} \mathcal{E}_n^p(\sigma, \sigma) = \infty$ for every $p > \frac{d-1}{d+1}$, if $k \leq d+1$, and transient if $k > d+1$.*

Note that this implies the same recurrence/transience classification for the $(d-1)$ -random walk from [PR12].

Our last result concerns the Dirichlet problem on simplicial complexes. Recall that for a finite graph $G = (V, E)$, $\emptyset \neq A \subset V$ and $f : A \rightarrow \mathbb{R}$ the unique solution to the Dirichlet problem¹, can be written using the random walk as $F(v) = E^v[f(Y_\tau)]$, where $(Y_n)_{n \geq 0}$ is the simple random walk on G and $\tau = \inf\{k \geq 0 : Y_k \in A\}$.

In Section 5, we discuss the high-dimensional analogue of the Dirichlet problem and show the following:

Theorem. *For every finite complex X , every subset A of the $(d-1)$ -simplexes satisfying a certain homological condition and every form f on A , there exists a unique solution to the Dirichlet problem that can be expressed in terms of the ESBRW.*

1.3 Structure of the paper

The remainder of this paper is organized as follows:

In Section 2 we introduce the relevant notation and definitions regarding: simplicial complexes (Subsection 2.1), high-dimensional Laplacians (Subsection 2.2), discrete Hodge theory (Subsection 2.3) and the $(d-1)$ -walk (Subsection 2.4).

In Section 3, we define the SBRW and the ESBRW, discuss some of their basic properties and prove the first two main results, Theorem 3.1 and Theorem 3.6.

Section 4 deals with application of the ESBRW to the study of arboreal complexes and provides the proof of Theorem 1.1 and Corollary 1.2.

In Section 5, the high-dimensional Dirichlet problem is discussed, and in particular how the ESBRW can be used to construct its solutions.

Section 6 explains how to construct a similar particle process corresponding to the lower Laplacian, thus allowing to generalize most of the results from previous sections to this setting.

The appendix provides the proof of some claims stated throughout the manuscript.

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¹The Dirichlet problem for a given triplet (G, A, f) is to find a solution $F : V \rightarrow \mathbb{R}$ to the boundary value problem $\Delta F = 0$ on $V \setminus A$ and $F = f$ on A , where Δ is the graph Laplacian.

2 Notation and some useful facts

This Section collects definitions, notation and previously known results used in the paper. Notion related to simplicial complexes can be found in Subsection 2.1. The definition of the high-dimensional Laplacians and the boundary/coboundary operators appear in Subsection 2.2. A short summary on discrete Hodge theory is the content of Subsection 2.3. Finally, Subsection 2.4 recalls the definition of the $(d-1)$ -walk from [PR12] as well as some of the results proved there regarding its connection to spectral and homological properties of the complex.

2.1 Simplicial complexes

A simplicial complex X is a collection of subsets of some countable set V that is closed under the operation of taking subsets. That is, if $\tau \in X$ and $\sigma \subset \tau$ then $\sigma \in X$. Elements of X are called simplexes or cells and the dimension of a simplex $\sigma \in X$ is defined to be $|\sigma| - 1$. A j -dimensional simplex is called a j -simplex or a j -cell. The dimension of X , denoted by d , is defined to be $\max_{\sigma \in X} \dim(\sigma)$. A d -dimensional simplicial complex is called a d -complex for short. We denote by X^j the set of j -dimensional cells. The degree of a j -cell, denoted $\deg(\sigma)$, is the number of $(j+1)$ -cells containing it and the set of such $(j+1)$ -cells, also known as its cofaces, is denoted by $\text{cf}(\sigma) = \{\tau \in X^{j+1} : \sigma \subset \tau\}$.

For $j \geq 1$, each j -cell has two possible orientation, corresponding to the ordering of its vertices up to an even permutation. Oriented cells are denoted by square brackets; for example, the unoriented 2-cell $\{u, v, w\}$ has two orientation $[u, v, w] = [w, u, v] = [v, w, u]$ and $[v, u, w] = [w, v, u] = [u, w, v]$. Given an oriented cell σ we denote by $\bar{\sigma}$ or $(-1)\sigma$ the same cell with the opposite orientation. The set of all oriented j cells is denoted by X_{\pm}^j . We also denote $X_{\pm}^j = X^j$ for $j = -1, 0$. The faces of a j -cell $\sigma = \{v_0, \dots, v_j\}$, abbreviated $\text{face}(\sigma)$, are the $(j-1)$ -cells $\{\sigma \setminus v_i\}_{i=0}^j$. An oriented j -cell $\sigma = [v_0, \dots, v_j]$, $j \geq 2$ induces an orientation on its faces given by $\left\{(-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_j]\right\}_{i=0}^j$. In a similar manner an oriented j -cell σ induces an orientation on its co-faces as follows: Given a cell σ and a vertex $v \notin \sigma$ such that $v\sigma := \{v\} \cup \sigma$ is a coface of σ we write shortly $v \triangleleft \sigma$. If $\sigma = [\sigma_0, \dots, \sigma_k]$ is oriented and $v \triangleleft \sigma$, then $v\sigma$ inherits the orientation $[v, \sigma_0, \dots, \sigma_k]$.

The space of j -forms on X , denoted $\Omega^j = \Omega^j(X)$, contains all function from X_{\pm}^j to \mathbb{R} which are anti-symmetric with respect to a change of orientation, namely

$$\Omega^j = \left\{ f : X_{\pm}^j \rightarrow \mathbb{R} \mid f(\bar{\sigma}) = -f(\sigma) \quad \forall \sigma \in X_{\pm}^j \right\}.$$

For $j = -1, 0$ there are no orientations and thus Ω^0 can be identified with the space of functions on the vertices, while Ω^{-1} can naturally be identified with \mathbb{R} . To every $\sigma \in X_{\pm}^j$ one can associate a Dirac j -form $\mathbb{1}_{\sigma}$ defined by

$$\mathbb{1}_{\sigma}(\sigma') = \begin{cases} 1 & \sigma' = \sigma \\ -1 & \sigma' = \bar{\sigma} \\ 0 & \text{otherwise} \end{cases}.$$

We also recall the following definitions from [PR12]:

Definition 2.1 (Simplices neighboring relation [PR12, Definition 2.1]). Let $1 \leq j \leq d-1$. Two oriented j -cells $\sigma, \sigma' \in X_{\pm}^j$ are called *neighbors* (denoted $\sigma \sim \sigma'$ or $\sigma \overset{\sim}{\sim} \sigma'$) if $\sigma \cup \sigma'$ is a $(j+1)$ -cell and the orientation induced by σ on $\sigma \cup \sigma'$ is opposite to the one induced on it by σ' . In the case $j \geq 2$ this is also equivalent to the assumption that $\sigma \cup \sigma' \in X^{j+1}$, and that the $(j-1)$ -cell $\sigma \cap \sigma'$ inherits the same orientation from both σ and σ' . In the case $j = 0$ the relation \sim is used to denote the usual graph neighboring relation, that is $\sigma \sim \sigma'$ if both 0-cells (vertices) are part of a common 1-cell (an edge).

Definition 2.2 (k -connectedness and disorientability [PR12, Definitions 2.2, 2.6]).

- (1) Let $0 \leq k \leq d-1$. We say that X is k -connected if for every pair of oriented k -cells σ, σ' there exists a chain $\sigma = \sigma_0 \sim \sigma_1 \sim \dots \sim \sigma_n = \sigma'$. Moreover, the existence of such a chain defines an equivalence relation on the k -cells of X , whose equivalence classes are called the k -components of X .
- (2) Let $0 \leq k \leq d-1$. A k -disorientation of a d -complex is a choice of orientation X_+^{k+1} for its $(k+1)$ -cells, so that whenever $\sigma, \sigma' \in X_+^{k+1}$ intersect in a k -cell they induce the same orientation on it. If X has a k -disorientation it is said to be k -diorientable.

A corresponding neighboring relation using faces instead of cofaces was defined in [MS13].

Definition 2.3 (Simplices adjacency relation [MS13, Definition 3.1]). Let $2 \leq j \leq d$. Two oriented j -cells $\sigma, \sigma' \in X_\pm^j$ are called adjacent (denoted $\sigma \sim_\downarrow \sigma'$) if $\sigma \cap \sigma'$ is a $(j-1)$ -cell that inherits opposite orientations from σ and σ' . If $\sigma \cup \sigma' \in X^{j+1}$ this is equivalent to saying that they induce the same orientation on their joint coface. In the case $j = 1$, two oriented 1-cells (edges) $\sigma, \sigma' \in X_\pm^1$ are called adjacent if $\sigma \cap \sigma'$ is a vertex and exactly one of the edges points towards it.

2.2 High dimensional Laplacians

Let X be a d -complex and $0 \leq k \leq d$. The k^{th} coboundary operator $\delta_k : \Omega^{k-1} \rightarrow \Omega^k$ is defined by

$$\delta_k f(\sigma) = \sum_{i=0}^k (-1)^i f(\sigma \setminus \sigma_i), \quad \forall f \in \Omega^k.$$

Given a weight function $w : X \rightarrow (0, \infty)$, one can introduce the Hilbert spaces

$$\Omega_{L^2}^k = \Omega_{L^2}^k(X) = \left\{ f \in \Omega^k(X) : \langle f, f \rangle < \infty \right\},$$

with inner product

$$\langle f, g \rangle = \sum_{\sigma \in X^k} w(\sigma) f(\sigma) g(\sigma), \quad \forall f, g \in \Omega^k.$$

Note that the sum is over unoriented k -cells, and that it is well defined since the product $f(\sigma) g(\sigma)$ is independent of the orientation.

Claim 2.4. Given a weight function $w : X \rightarrow (0, \infty)$. The operator δ_k is bounded if and only if $\sup_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) < \infty$.

Whenever δ_k is bounded its adjoint $\partial_k := \delta_k^* : \Omega_{L^2}^k \rightarrow \Omega_{L^2}^{k-1}$ is defined by the relation $\langle \delta_k f, g \rangle = \langle f, \partial_k g \rangle$ for every $f \in \Omega_{L^2}^{k-1}$ and every $g \in \Omega_{L^2}^k$. One can verify that in this case

$$\partial_k g(\sigma) = \frac{1}{w(\sigma)} \sum_{v \triangleleft \sigma} w(v\sigma) g(v\sigma). \quad (2.1)$$

The last equation can be taken as the definition of ∂_k even when the required assumptions on δ_k are not satisfied, however in this case $\partial_k : \Omega_{L^2}^k \rightarrow \Omega^{k-1}$ is not necessarily well defined since $\deg(\sigma)$ might be infinite.

Claim 2.5. If $\deg(\sigma) < \infty$ for every $\sigma \in X^{k-1}$ then ∂_k is well defined. In addition, the operator ∂_k is bounded whenever $\sup_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) < \infty$.

The last two claims portend the following definition:

Definition 2.6. A weight function $w : X \rightarrow (0, \infty)$ is called k -good if

$$\sup_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) < \infty. \quad (2.2)$$

If w is k -good for every $0 \leq k \leq d$ we simply say that w is good.

Example 2.7.

- (1) If X is a finite d -complex, then every weight function $w : X \rightarrow (0, \infty)$ is good.
- (2) Assume that X is a d -complex such that $1 \leq \deg(\sigma) < \infty$ for every $\sigma \in X^{d-1}$ and let $w_\uparrow : X \rightarrow (0, \infty)$ be the weight function

$$w_\uparrow(\sigma) = \begin{cases} \deg(\sigma) & \sigma \in X^{d-1} \\ 1 & \sigma \notin X^{d-1} \end{cases}.$$

Then for $\sigma \in X^{k-1}$

$$\frac{1}{w_\uparrow(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w_\uparrow(\tau) = \begin{cases} 1 & k = d \\ \sum_{\tau \in \text{cf}(\sigma)} \deg(\tau) & k = d - 1 \\ \deg(\sigma) & k < d - 1 \end{cases}.$$

Therefore w_\uparrow is always d -good, is $(d-1)$ -good if and only if $\sup_{\sigma \in X^{d-2}} \sum_{\tau \in \text{cf}(\sigma)} \deg(\tau) < \infty$ and is k -good for $k < d-1$ if and only if the degrees of the $(k-1)$ -cells are uniformly bounded.

- (3) Assume that X is a d -complex such that $\deg(\sigma) < \infty$ for every $\sigma \in X^{d-1}$ and let $w_\downarrow : X \rightarrow (0, \infty)$ be the weight function

$$w_\downarrow(\sigma) = \begin{cases} \frac{1}{d+1} & \sigma \in X^d \\ 1 & \sigma \notin X^d \end{cases}.$$

Then for $\sigma \in X^{k-1}$

$$\frac{1}{w_\downarrow(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w_\downarrow(\tau) = \begin{cases} \frac{\deg(\sigma)}{d+1} & k = d \\ \deg(\sigma) & k < d \end{cases}.$$

Therefore w_\downarrow is k -good if and only if the degrees of the $(k-1)$ -cells are uniformly bounded.

Whenever the operators ∂ . and δ . are well defined, the *upper, lower and full Laplacians*, $\Delta_k^+, \Delta_k^- : \Omega_{L^2}^k \rightarrow \Omega^k$ and $\Delta_k : \Omega_{L^2}^k \rightarrow \Omega^k$ respectively, are given by

$$\begin{aligned} \Delta_k^+ &= \partial_{k+1} \delta_{k+1} & -1 \leq k \leq d-1, \\ \Delta_k^- &= \delta_k \partial_k & 0 \leq k \leq d, \\ \Delta_k &= \Delta_k^+ + \Delta_k^- & 0 \leq k \leq d-1. \end{aligned}$$

The special case of Δ_{d-1}^+ will be abbreviated Δ^+ .

A short calculation gives

$$\Delta_k^+ f(\sigma) = \frac{1}{w(\sigma)} \left(\sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) f(\sigma) - \frac{1}{w(\sigma)} \sum_{\sigma' \sim \sigma} w(\sigma' \cup \sigma) f(\sigma') \quad (2.3)$$

and

$$\Delta_k^- f(\sigma) = \left(\sum_{\tau \in \text{face}(\sigma)} \frac{w(\sigma)}{w(\tau)} \right) f(\sigma) - \sum_{\substack{\sigma' \sim \sigma \\ \downarrow}} \frac{w(\sigma')}{w(\sigma \cap \sigma')} f(\sigma'). \quad (2.4)$$

The space of *Harmonic k -forms*, denoted $\mathcal{H}^k = \mathcal{H}^k(X)$, is defined to be the kernel of Δ_k .

Throughout the paper (except for Section 6) the weight function w_\uparrow from Example 2.7(2) is used, in which case one gets

$$\Delta^+ f(\sigma) = f(\sigma) - \frac{1}{\deg(\sigma)} \sum_{\sigma' \sim \sigma} f(\sigma'). \quad (2.5)$$

2.3 Discrete Hodge theory

The sequence (Ω^k, δ_{k+1}) is a simplicial cochain complex of X , meaning that $\delta_{k+1}\delta_k = 0$ for every k . The chain structure gives rise to the

$$\begin{array}{ll} k\text{-cocycles (closed forms)} & Z^k = \ker \delta_{k+1}, \\ k\text{-coboundaries (exact forms)} & B^k = \text{im} \delta_k, \\ k\text{-cohomology} & H^k = Z^k / B^k. \end{array}$$

When X is a **finite** complex (or more generally when w is a good weight function) the sequence (Ω^k, ∂_k) is a simplicial chain complex of X and this gives rise to

$$\begin{array}{ll} k\text{-cycles} & Z_k = \ker \partial_k, \\ k\text{-boundaries} & B_k = \text{im} \partial_{k+1}, \\ k\text{-homology} & H_k = Z_k / B_k. \end{array}$$

The isomorphism between harmonic k -forms, the k -cohomology and the k -homology as well as the connection to the boundary operators is known as discrete Hodge theorem. In the discrete setting it originates in the Work of Eckmann [Eck45] and is summarized in the following lemma:

Lemma 2.8 (Discrete Hodge theory [Eck45]). *Let X be a **finite** d -complex. Then for any $-1 \leq k \leq d-1$ and any weight function $w : X \rightarrow (0, \infty)$*

- (1) $Z_k = \ker \partial_k = \ker \Delta_k^- = (B^k)^\perp$.
- (2) $B_k = \text{im} \partial_{k+1} = \text{im} \Delta_k^+ = (Z^k)^\perp$.
- (3) $Z^k = \ker \delta_{k+1} = \ker \Delta_k^+ = (B_k)^\perp$.
- (4) $B^k = \text{im} \delta_k = \text{im} \Delta_k^- = (Z_k)^\perp$.
- (5) $\mathcal{H}^k = \ker \Delta_k = Z_k \cap Z^k = (B_k \oplus B^k)^\perp \cong H_k \cong H^k$.
- (6) $\Omega^k = \overbrace{B_k \oplus \mathcal{H}^k}^{Z_k} \oplus \underbrace{B^k}_{Z^k}$ (Hodge decomposition).

Due to the chain complex structure $B_k \subset Z_k$, which implies that the Laplacian Δ_k^+ always has trivial zeroes in its kernel. The *spectral gap* of a finite d -complex X , denoted $\lambda_k(X)$, is defined to be the smallest non-trivial eigenvalue of Δ_k^+ that is

$$\lambda_k(X) = \min \left(\text{Spec} \left(\Delta_k^+ \Big|_{(B^{k-1})^\perp} \right) \right) = \min \left(\text{Spec} \left(\Delta_k^+ \Big|_{Z_{k-1}} \right) \right).$$

Going back to the case of a general d -complex X , the boundary operators ∂_k are not well defined and even when they are it is possible to have $\partial_k \partial_{k+1}(f) \neq 0$ for some $f \in \Omega_{L^2}^{k+1}$. If $w : X \rightarrow (0, \infty)$ is both k and $(k+1)$ -good then both operators are well defined and bounded and due to the fact that $\partial_k^* = \delta_k$ and $\partial_{k+1} = \delta_{k+1}^*$ it follows that $\partial_k \partial_{k+1} = 0$. The interested reader might want to consult [PR12] for additional discussion on the general case.

2.4 The $(d-1)$ -walk

In this subsection we recall the definition of the $(d-1)$ -walk constructed in [PR12] as well as some of its properties.

Definition 2.9. [PR12, Definition 2.1] The p -lazy $(d-1)$ -walk on X is a time-homogeneous Markov chain with state space X_{\pm}^{d-1} that stays put with probability p , and with probability $(1-p)$ chooses one of its neighbors (see Definition 2.1) in X_{\pm}^{d-1} uniformly at random and jumps to it. More formally, this is a Markov chain $(Y_n)_{n \geq 0}$ with state space X_{\pm}^{d-1} and transition probabilities

$$\text{Prob} \left(Y_{n+1} = \sigma' \middle| Y_n = \sigma \right) = \begin{cases} p & \sigma' = \sigma \\ \frac{1-p}{d \cdot \deg(\sigma)} & \sigma' \sim \sigma \\ 0 & \text{otherwise} \end{cases}.$$

The heat kernel of the random walk $(\mathbf{p}_n(\sigma, \sigma'))_{n \geq 0, \sigma, \sigma' \in X_{\pm}^{d-1}}$ is defined by

$$\mathbf{p}_n(\sigma, \sigma') = \text{Prob} \left(Y_n = \sigma' \middle| Y_0 = \sigma \right).$$

The behavior of the $(d-1)$ -random walk, or more precisely of its heat kernel, relates to the $(d-1)$ -connectedness of the complex in the same way that a classic random walk on a graph relates to the connectedness of the graph. In order to relate the $(d-1)$ -walk to the homology and cohomology of the complex, which are more natural counterparts of connectedness in high dimensions, the authors introduced the *expectation process* $\mathcal{E}_n : X_{\pm}^{d-1} \times X_{\pm}^{d-1} \rightarrow [-1, 1]$ which for $d \geq 2$ is defined by²

$$\mathcal{E}_n(\sigma, \sigma') = \mathbf{p}_n(\sigma, \sigma') - \mathbf{p}_n(\sigma, \overline{\sigma'}).$$

Unfortunately a new problem arises when observing the expectation process, that is $\lim_{n \rightarrow \infty} \mathcal{E}_n(\sigma, \sigma') = 0$, for every $\sigma, \sigma' \in X_{\pm}^{d-1}$. However, it was proven in [PR12] that $|\mathcal{E}_n(\sigma, \sigma')| = \Theta \left(\left(\frac{p(d-1)+1}{d} \right)^n \right)$ for every finite d -complex X , which called upon the definition of a *normalized expectation process*

$$\tilde{\mathcal{E}}_n(\sigma, \sigma') = \left(\frac{d}{p(d-1)+1} \right)^n \mathcal{E}_n(\sigma, \sigma').$$

The evolution of the expectation process and its normalized version in time is given by $\mathcal{E}_{n+1}(\sigma, \cdot) = (A_p \mathcal{E}_n)(\sigma, \cdot)$ and $\tilde{\mathcal{E}}_{n+1}(\sigma, \cdot) = \left(\left(\frac{d}{p(d-1)+1} \right) A_p \tilde{\mathcal{E}}_n \right)(\sigma, \cdot)$ respectively, where

$$(A_p f)(\sigma) = pf(\sigma) + \frac{(1-p)}{d} \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')}, \quad (2.6)$$

and A_p acts on the second coordinate.

The following theorem summarizes the connection between the asymptotics of the normalized expectation process and the homology of the a complex:

Theorem 2.10 ([PR12, Theorem 2.9 and (2.1)]). *Let X be a **finite** d -complex and $\tilde{\mathcal{E}}_n$ the normalized expectation process associated with the p -lazy $(d-1)$ -walk on X .*

(1) *If $\frac{d-1}{3d-1} < p < 1$, then $\tilde{\mathcal{E}}_{\infty} = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n$ always exists. In addition if $p = \frac{d-1}{3d-1}$ then $\tilde{\mathcal{E}}_{\infty} = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n$ exists whenever X has no disorientable $(d-1)$ -components.*

(2) *If $p > \frac{d-1}{3d-1}$ or $p = \frac{d-1}{3d-1}$ and X has no disorientable $(d-1)$ -components, then $\left\{ \tilde{\mathcal{E}}_{\infty}(\sigma, \cdot) \right\}_{\sigma \in X_{\pm}^{d-1}} \subset B^{d-1}$ if and only if $H_{d-1}(X) = 0$.*

²In the case $d = 1$ the expectation process is simply defined to be heat kernel.

- (3) More generally, If $p > \frac{d-1}{3d-1}$ or $p = \frac{d-1}{3d-1}$ and X has no disorientable $(d-1)$ -components, then the dimension of $H_{d-1}(X)$ equals the dimension of $\text{Span} \left\{ \text{Proj}_{Z_{d-1}} \left(\tilde{\mathcal{E}}_\infty(\sigma, \cdot) \right) : \sigma \in X_\pm^{d-1} \right\}$, where $\text{Proj}_{Z_{d-1}}$ is the orthogonal projection in Ω^{d-1} onto Z_{d-1} .
- (4) If furthermore $p \geq \frac{1}{2}$ then

$$\text{dist} \left(\tilde{\mathcal{E}}_n, B^{d-1} \right) = O \left(\left(1 - \frac{1-p}{p(d-1)+1} \lambda_{d-1}(X) \right)^n \right).$$

Remark 2.11. When necessary the notation \mathcal{E}_n^p and $\tilde{\mathcal{E}}_n^p$ is used to stress the dependence of \mathcal{E}_n and $\tilde{\mathcal{E}}_n$ on p .

3 Simplicial branching random walks

This section is devoted to the definition of simplicial branching random walk and its effective version as well as to the study of their basic properties. In the first part, the definition of the processes is given and the first result (Theorem 3.1) is proved. In the second part, the associated tree structure is described and a high-dimensional version of (1.1) is proved (see Theorem 3.6). A discussion on several possible variations of the model can be found in Remark 3.3. Throughout this section X denotes a d -complex such that $1 \leq \deg(\sigma) < \infty$ for every $\sigma \in X^{d-1}$.

The p -lazy simplicial branching random walk on X is a time-homogeneous Markov chain $(N_n(\cdot))_{n \geq 0}$ with state space $\mathbb{N}^{X_\pm^{d-1}}$ which describes the number of particles at time n on any of the oriented $(d-1)$ -cells, that is:

- N_n is a random function from X_\pm^{d-1} to \mathbb{N} .
- The process is Markovian, i.e., $\text{Prob} \left(N_n \in A \mid N_1, \dots, N_{n-1} \right) = \text{Prob} \left(N_n \in A \mid N_{n-1} \right)$ and time homogeneous, namely $\text{Prob} \left(N_n = g \mid N_{n-1} = f \right)$ doesn't depend on n .
- $N_n(\sigma)$ is the random number of particles in σ at time n for every $\sigma \in X_\pm^{d-1}$ and $n \geq 0$.

One step evolution of the process (its transition kernel) is defined as follows: Given a configuration of particles on X_\pm^{d-1} all the particles evolve simultaneously and independently. If a particle is positioned in σ , then it stays put with probability p , and with probability $1-p$ chooses one of the cofaces of σ uniformly at random and splits into d new particles which are now positioned on the neighbors of σ in the chosen coface (one on each such neighbor). Note that one step of the process is comprised of the evolution of all existing particles. An illustration of one step of the process on a triangle complex can be found in Figure 1.1.

One way to realize the process is as follows: Let $(\eta_\sigma)_{\sigma \in X_\pm^{d-1}}$ be random variables taking values in $X^d \uplus \{\zeta\}$, with each η_σ distributed like

$$\text{Prob}(\eta_\sigma = \tau) = \begin{cases} p & , \tau = \zeta \\ \frac{1-p}{\deg(\sigma)} & , \tau \in \text{cf}(\sigma) \end{cases}.$$

Then $(N_n)_{n \geq 0}$ is a Markovian process, taking values in $\mathbb{N}^{X_\pm^{d-1}}$ defined by

$$N_{n+1}(\sigma) = \sum_{i=1}^{N_n(\sigma)} \mathbb{1}_{\zeta_\sigma^{i,n} = \zeta} + \sum_{\sigma' \sim \sigma} \sum_{i=1}^{N_n(\sigma')} \mathbb{1}_{\zeta_\sigma^{i,n} = \sigma \cup \sigma'},$$

where $(\eta_\sigma^{i,n})_{i \geq 1, n \geq 0}$ are i.i.d. copies of η_σ .

For a given $\pi \in \mathbb{N}^{X_\pm^{d-1}}$ we denote by P^π the distribution of $(N_n)_{n \geq 0}$ with the above law and starting distribution $P^\pi(N_0 = \pi) = 1$. The expectation with respect to P^π is denoted by E^π . In the case $\pi = \delta_\sigma$, where $\delta_\sigma(\sigma') = \begin{cases} 1 & \sigma' = \sigma \\ 0 & \text{otherwise} \end{cases}$, we abbreviate P^σ and E^σ instead of P^{δ_σ} and E^{δ_σ} .

The process which is truly the source of our interest is not $(N_n)_{n \geq 0}$ but rather its effective version defined by

$$D_n(\sigma) = N_n(\sigma) - N_n(\bar{\sigma}), \quad \forall \sigma \in X_\pm^{d-1}. \quad (3.1)$$

Note that $(D_n)_{n \geq 0}$ is a sequence of random forms in Ω^{d-1} .

Finally, the heat kernel of $(D_n)_{n \geq 0}$ is defined by

$$\mathcal{E}_n(\sigma, \sigma') = E^\sigma[D_n(\sigma')], \quad \forall \sigma, \sigma' \in X_\pm^{d-1}.$$

When necessary the notation $\mathcal{E}_n^p(\sigma, \sigma')$ will be used to stress the dependence on p .

We are now ready to give a formal statement of our first result:

Theorem 3.1. *Let X be a **finite** d -complex, $(D_n)_{n \geq 0}$ the p -lazy ESBRW on X and $(\mathcal{E}_n)_{n \geq 0}$ its heat kernel.*

- (1) *If $\frac{d-1}{d+1} < p < 1$, then $\mathcal{E}_\infty = \lim_{n \rightarrow \infty} \mathcal{E}_n$ always exists. In addition if $p = \frac{d-1}{d+1}$ then $\mathcal{E}_\infty = \lim_{n \rightarrow \infty} \mathcal{E}_n$ exists whenever X has no disorientable $(d-1)$ -components.*
- (2) *If $p > \frac{d-1}{d+1}$ or $p = \frac{d-1}{d+1}$ and X has no disorientable $(d-1)$ -components, then $\{\mathcal{E}_\infty(\sigma, \cdot)\}_{\sigma \in X_\pm^{d-1}} \subset B^{d-1}$ if and only if $H_{d-1}(X) = 0$.*
- (3) *More generally, if $p > \frac{d-1}{d+1}$ or $p = \frac{d-1}{d+1}$ and X has no disorientable $(d-1)$ -components, then the dimension of $H_{d-1}(X)$ equals the dimension of $\text{Span}\left\{\text{Proj}_{Z_{d-1}}(\mathcal{E}_\infty(\sigma, \cdot)) : \sigma \in X_\pm^{d-1}\right\}$, where $\text{Proj}_{Z_{d-1}}$ is the orthogonal projection in Ω^{d-1} onto Z_{d-1} .*
- (4) *If furthermore $p \geq \frac{d}{d+1}$ then*

$$\text{dist}(\mathcal{E}_n, B^{d-1}) = O((1 - (1-p)\lambda_{d-1}(X))^n).$$

The following lemma contains the main ingredient for the proof of Theorem 3.1 besides Theorem 2.10:

Lemma 3.2 (Time evolution of the heat kernel and its connection to the expectation process).

- (1) *For every $0 \leq p \leq 1$ and $\sigma' \in X_\pm^{d-1}$ the evolution of $\mathcal{E}_n^p(\cdot, \sigma')$ in time is given by $\mathcal{E}_n^p(\cdot, \sigma') = (\mathcal{A}_p \mathcal{E}_{n-1}^p)(\cdot, \sigma')$, where $\mathcal{A}_p : \Omega^{d-1} \rightarrow \Omega^{d-1}$ acts on the first coordinate and is given by*

$$\mathcal{A}_p f(\sigma) = pf(\sigma) + \frac{1-p}{\deg(\sigma)} \sum_{\sigma' \sim \sigma} f(\sigma') = (I - (1-p)\Delta^+)f(\sigma), \quad \forall f \in \Omega_{L^2}^{d-1}.$$

- (2) *For every $0 \leq p \leq 1$*

$$\mathcal{E}_n^p(\sigma, \sigma') = \tilde{\mathcal{E}}_n^{p'}(\sigma, \sigma'), \quad \forall \sigma, \sigma' \in X_\pm^{d-1},$$

$$\text{where } p' = \frac{p}{1+(1-p)(d-1)}.$$

Proof.

(1) By the Markov property, for every $n \geq 1$

$$\begin{aligned}
\mathcal{E}_n(\sigma, \sigma') &= E^\sigma \left[E^\sigma \left[D_n(\sigma') \mid N_1 \right] \right] = E^\sigma \left[E^{N_1} \left[D_{n-1}(\sigma') \right] \right] \\
&= E^\sigma \left[\sum_{\sigma'' \in X_{\pm}^{d-1}} N_1(\sigma'') \cdot E^{\sigma''} \left[D_{n-1}(\sigma') \right] \right] = \sum_{\sigma'' \in X_{\pm}^{d-1}} E^\sigma \left[N_1(\sigma'') \right] \mathcal{E}_{n-1}(\sigma'', \sigma') \\
&= p \mathcal{E}_{n-1}(\sigma, \sigma') + \frac{1-p}{\deg(\sigma)} \cdot \sum_{\sigma'' \sim \sigma} \mathcal{E}_{n-1}(\sigma'', \sigma'). \tag{3.2}
\end{aligned}$$

(2) Using part (1) and the equality $\mathcal{E}_0(\sigma, \sigma') = \mathbb{1}_\sigma(\sigma')$ it follows that $\mathcal{E}_n^p(\sigma, \sigma') = \deg(\sigma') \cdot \langle \mathcal{A}_p^n \mathbb{1}_\sigma, \mathbb{1}_{\sigma'} \rangle$ for every $n \geq 0$. Similarly, by (2.6)

$$\begin{aligned}
\tilde{\mathcal{E}}_n^{p'}(\sigma, \sigma') &= \left(\frac{d}{p'(d-1)+1} \right)^n \mathcal{E}_n^{p'}(\sigma, \sigma') = \deg(\sigma') \cdot \left(\frac{d}{p'(d-1)+1} \right)^n \langle \mathbb{1}_\sigma, A_{p'}^n \mathbb{1}_{\sigma'} \rangle \\
&= \deg(\sigma') \cdot \left\langle \left(\left(\frac{d}{p'(d-1)+1} \right) \cdot A_{p'}^{tr} \right)^n \mathbb{1}_\sigma, \mathbb{1}_{\sigma'} \right\rangle
\end{aligned}$$

for every $n \geq 0$, where A_p^{tr} is the transpose of A_p . The claim now follows since

$$\mathcal{A}_p = I - (1-p) \Delta^+ = \left(\frac{d}{p'(d-1)+1} \right) \cdot A_{p'}^{tr},$$

$$\text{for } p' = \frac{p}{1+(1-p)(d-1)}.$$

□

Proof of Theorem 3.1. The proof follows by combining Theorem 2.10 and Lemma 3.2(2). □

Remark 3.3 (Variants on the model). Before turning to discuss the tree structure associated with the SBRW we wish to introduce some possible variants for the model. First, due to the fact that our main interest lies in the ESBRW $(D_n)_{n \geq 0}$ and not in the SBRW itself, it is possible to annihilate any pair of particles on the same cell with different orientation. That is, if at time n there are $N_n(\sigma) = k_1$ and $N_n(\bar{\sigma}) = k_2$ particles of type σ and $\bar{\sigma}$ respectively, and without loss of generality $k_1 \geq k_2$, then all of them annihilates except for $k_1 - k_2$ of the σ particles. This variant on the model is nothing else than a different choice of coupling for the branching of the particles. Indeed, for this choice any pair of particles on the same cell with different orientation are coupled to branch together. Other couplings can also be considered. Secondly, in order to avoid simultaneous splitting of the particles one can work with a continuous time version where each particle has a Poisson clock to determine its branching time. Finally, note that the above model can also be generalized to give a high-dimensional analogue of weighted random walk on graphs by considering other weight functions.

In addition, one can also consider the above process on k -oriented cells of a d -complex for every $0 \leq k \leq d-1$ and not just for $k = d-1$. This however is equivalent to studying the original process on X^{k+1} and therefore falls back to the above setting.

3.1 Expected number of first visits

The SBRW is a fusion between a multi-type branching process and a random walk. On the one hand in every step the current population of particles splits and creates a new population in the same manner as in a branching process. On the other hand the law that specify the siblings of each particle is governed by the law of a $(d-1)$ -random walk on the complex.

To every branching process, and in particular the SBRW, one can associate a natural tree structure where the siblings of each particle are the one generated from it (see Figure 3.1 for an illustration). The tree structure also allows us to associate with each particle a unique sequence of ancestors. These facts are summarized in the following definition:

Definition 3.4.

- (1) For $\sigma \in X_{\pm}^{d-1}$ and $n \geq 0$ let $\Psi_n(\sigma)$ be the set of particles in σ at time n . Note that $N_n(\sigma) = |\Psi_n(\sigma)|$.
- (2) Denote $\Psi_n = \bigsqcup_{\sigma \in X_{\pm}^{d-1}} \Psi_n(\sigma)$, the set of all particles at time n .
- (3) To each element in the set of particles at time n one can associate a unique sequence of ancestors going back to the set of particles at time 0. Given $\xi \in \Psi_n$ denote by $\mathbf{a}\xi$ the unique ancestor of ξ in Ψ_{n-1} . Continuing recursively one can define $\mathbf{a}^k \xi = \mathbf{a}(\mathbf{a}^{k-1} \xi)$ for every $2 \leq k \leq n$.

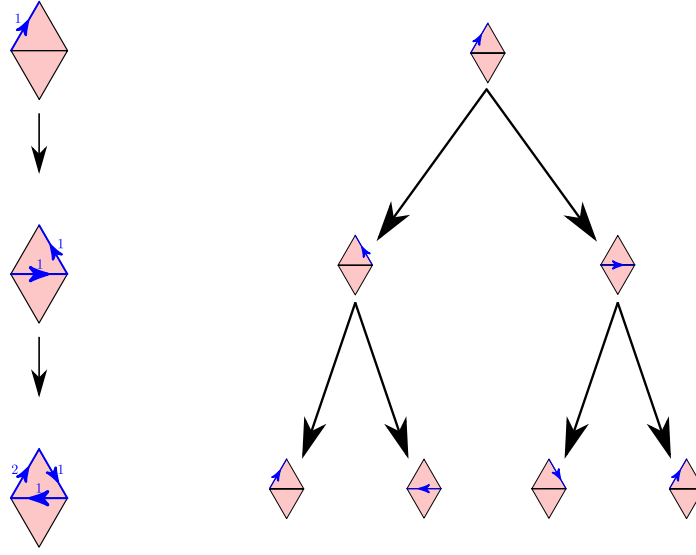


Figure 3.1: Two steps of the branching random walk and the tree associated with it.

The definition of ancestors of a particle allows us to generalize the important notion of first return to a vertex from random walks on graphs:

Definition 3.5. For $\sigma \in X_{\pm}^{d-1}$ and $n \geq 1$ define $K_n(\sigma)$ to be the number of particles in σ at time n that none of their ancestors (except perhaps to the one at time zero) were in σ or $\bar{\sigma}$. Namely,

$$K_n(\sigma) = \# \left\{ \xi \in \Psi_n(\sigma) : \mathbf{a}^k \xi \notin \Psi_{n-k}(\sigma) \cup \Psi_{n-k}(\bar{\sigma}) \quad \forall 1 \leq k < n \right\}.$$

For $n = 0$ define $K_0(\sigma) = 0$. We also denote $F_n(\sigma) = K_n(\sigma) - K_n(\bar{\sigma})$ and $\mathcal{F}_n(\sigma, \sigma') = E^\sigma [F_n(\sigma')]$.

The main goal of this subsection is to prove the following result:

Theorem 3.6. Let X be a d -complex, $\sigma \in X_{\pm}^{d-1}$ and $z \in \mathbb{C}$. Define the power series

$$(1) \quad \mathcal{G}(z) = \sum_{n=0}^{\infty} \mathcal{E}_n(\sigma, \sigma) z^n$$

$$(2) \quad \mathfrak{F}(z) = \sum_{n=0}^{\infty} \mathcal{F}_n(\sigma, \sigma) z^n$$

Then, for every $z \in \mathbb{C}$ whose absolute value is smaller than the radii of convergence of both power series

$$\mathcal{G}(z) = \frac{1}{1 - \mathfrak{F}(z)}$$

as long as $\mathfrak{F}(z) \neq 1$.

Remark 3.7.

- (1) The radii of convergence of the above power series are at least $\frac{1}{d}$ since the definition of SBRW guarantees that

$$|\mathcal{E}_n(\sigma, \sigma)|, |\mathcal{F}_n(\sigma, \sigma)| \leq d^n, \quad \forall n \geq 0.$$

- (2) This is a high-dimensional analogue of (1.1).

The following lemma contains several connections between $(\mathcal{E}_n)_{n \geq 0}$ and $(\mathcal{F}_n)_{n \geq 0}$ which will be useful for the proof of Theorem 3.6:

Lemma 3.8. *The following relations hold for every $\sigma, \sigma' \in X_{\pm}^{d-1}$:*

$$(1) \mathcal{E}_n(\sigma, \sigma') = \sum_{\sigma'' \in X_{\pm}^{d-1}} E^{\sigma} [N_1(\sigma'')] \mathcal{E}_{n-1}(\sigma'', \sigma').$$

$$(2) E^{\sigma} [K_n(\sigma')] = \sum_{\sigma'' \in (X^{d-1} \setminus \sigma')_{\pm}} E^{\sigma} [N_1(\sigma'')] E^{\sigma''} [K_{n-1}(\sigma')] \text{ for } n \geq 2. \text{ In particular this gives:}$$

$$(a) \mathcal{F}_n(\sigma, \sigma') = \sum_{\sigma'' \in (X^{d-1} \setminus \sigma')_{\pm}} E^{\sigma} [N_1(\sigma'')] \mathcal{F}_{n-1}(\sigma'', \sigma') \text{ for } n \geq 2.$$

$$(b) E^{\sigma} [K_n(\sigma')] = \sum_{\sigma_1, \dots, \sigma_{n-1} \in (X^{d-1} \setminus \sigma')_{\pm}} E^{\sigma} [N_1(\sigma_1)] E^{\sigma_1} [N_1(\sigma_2)] \dots E^{\sigma_{n-2}} [N_1(\sigma_{n-1})] E^{\sigma_{n-1}} [N_1(\sigma')] \\ \text{for } n \geq 2, \text{ and } E^{\sigma} [K_1(\sigma')] = E^{\sigma} [N_1(\sigma')].$$

$$(3) \mathcal{E}_n(\sigma, \sigma') = \sum_{k=1}^n \mathcal{F}_k(\sigma, \sigma') \mathcal{E}_{n-k}(\sigma', \sigma') \text{ for } n \geq 1.$$

Proof.

- (1) The proof follows by the same argument as in Lemma 3.2(1).

- (2) Using the Markov property, for every $n \geq 2$

$$\begin{aligned} & E^{\sigma} [K_n(\sigma')] \\ &= E^{\sigma} \left[\# \left\{ \xi \in \Psi_n(\sigma') : \mathfrak{a}^k \xi \notin \Psi_{n-k}(\sigma') \cup \Psi_{n-k}(\overline{\sigma'}) \quad \forall 1 \leq k < n \right\} \right] \\ &= \sum_{\sigma'' \in (X^{d-1} \setminus \sigma')_{\pm}} E^{\sigma} [N_1(\sigma_1)] E^{\sigma''} \left[\# \left\{ \xi \in \Psi_{n-1}(\sigma') : \mathfrak{a}^k \xi \notin \Psi_{n-1-k}(\sigma') \cup \Psi_{n-1-k}(\overline{\sigma'}) \quad \forall 1 \leq k < n-1 \right\} \right] \\ &= \sum_{\sigma'' \in (X^{d-1} \setminus \sigma')_{\pm}} E^{\sigma} [N_1(\sigma'')] E^{\sigma''} [K_{n-1}(\sigma)]. \end{aligned}$$

(2)(a) follows from the fact that $\mathcal{F}_n(\sigma, \sigma') = E^{\sigma} [F_n(\sigma')] = E^{\sigma} [K_n(\sigma') - K_n(\overline{\sigma})]$. (2)(b) follows by induction using the fact that for $n = 1$:

$$\begin{aligned} E^{\sigma} [K_1(\sigma')] &= E^{\sigma} \left[\# \left\{ \xi \in \Psi_1(\sigma') : \mathfrak{a}^k \xi \notin \Psi_{1-k}(\sigma') \cup \Psi_{1-k}(\overline{\sigma'}) \quad \forall 1 \leq k < 1 \right\} \right] \\ &= E^{\sigma} [|\Psi_1(\sigma')|] = E^{\sigma} [N_1(\sigma')]. \end{aligned}$$

(3) The proof follows by induction and the Markov property. First note that for $n = 1$

$$\mathcal{E}_1(\sigma, \sigma') = E^\sigma [N_1(\sigma')] = \sum_{k=1}^1 \mathcal{F}_k(\sigma, \sigma') \mathcal{E}_{1-k}(\sigma', \sigma').$$

Assume next that the relation holds for $n - 1$, then by part (1)

$$\begin{aligned} \mathcal{E}_n(\sigma, \sigma') &= \sum_{\sigma'' \in X_{\pm}^{d-1}} E^\sigma [N_1(\sigma'')] \mathcal{E}_{n-1}(\sigma'', \sigma') \\ &= \sum_{\sigma'' \in X_{\pm}^{d-1}} E^\sigma [N_1(\sigma'')] \sum_{k=1}^{n-1} \mathcal{F}_k(\sigma'', \sigma') \mathcal{E}_{n-1-k}(\sigma', \sigma') \\ &= \sum_{k=1}^{n-1} \left(\sum_{\sigma'' \in X_{\pm}^{d-1}} E^\sigma [N_1(\sigma'')] \mathcal{F}_k(\sigma'', \sigma') \right) \mathcal{E}_{n-1-k}(\sigma', \sigma'). \end{aligned}$$

However by (2)(a)

$$\begin{aligned} \sum_{\sigma'' \in X_{\pm}^{d-1}} E^\sigma [N_1(\sigma'')] \mathcal{F}_k(\sigma'', \sigma') &= \sum_{\sigma'' \in (X^{d-1} \setminus \sigma')_{\pm}} E^\sigma [N_1(\sigma'')] \mathcal{F}_k(\sigma'', \sigma') \\ &\quad + E^\sigma [N_1(\sigma')] \mathcal{F}_k(\sigma', \sigma') + E^\sigma [N_1(\overline{\sigma}')] \mathcal{F}_k(\overline{\sigma}', \sigma') \\ &= \mathcal{F}_{k+1}(\sigma, \sigma') + (E^\sigma [N_1(\sigma')] - E^\sigma [N_1(\overline{\sigma}')]) \mathcal{F}_k(\sigma', \sigma') \\ &= \mathcal{F}_{k+1}(\sigma, \sigma') + \mathcal{F}_1(\sigma, \sigma') \mathcal{F}_k(\sigma', \sigma') \end{aligned}$$

and therefore by the induction hypothesis

$$\begin{aligned} \mathcal{E}_n(\sigma, \sigma') &= \sum_{k=1}^{n-1} \mathcal{F}_{k+1}(\sigma, \sigma') \mathcal{E}_{n-1-k}(\sigma', \sigma') + \sum_{k=1}^{n-1} \mathcal{F}_1(\sigma, \sigma') \mathcal{F}_k(\sigma', \sigma') \mathcal{E}_{n-1-k}(\sigma', \sigma') \\ &= \sum_{k=2}^n \mathcal{F}_k(\sigma, \sigma') \mathcal{E}_{n-k}(\sigma', \sigma') + \mathcal{F}_1(\sigma, \sigma') \cdot \sum_{k=1}^{n-1} \mathcal{F}_k(\sigma', \sigma') \mathcal{E}_{n-1-k}(\sigma', \sigma') \\ &= \sum_{k=2}^n \mathcal{F}_k(\sigma, \sigma') \mathcal{E}_{n-k}(\sigma', \sigma') + \mathcal{F}_1(\sigma, \sigma') \cdot \mathcal{E}_{n-1}(\sigma', \sigma') \\ &= \sum_{k=1}^n \mathcal{F}_k(\sigma, \sigma') \mathcal{E}_{n-k}(\sigma', \sigma'). \end{aligned}$$

□

Proof of Proposition 3.6. The statement will follow once we show that for every $z \in \mathbb{C}$ whose absolute value is smaller than the radii of convergence of both power series

$$\mathcal{G}(z) = 1 + \mathfrak{F}(z) \mathcal{G}(z).$$

This however follows from Lemma 3.8(3) since

$$\begin{aligned}
\mathcal{G}(z) &= \sum_{n=0}^{\infty} \mathcal{E}_n(\sigma_0, \sigma_0) z^n = 1 + \sum_{n=1}^{\infty} \mathcal{E}_n(\sigma_0, \sigma_0) z^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \mathcal{F}_k(\sigma_0, \sigma_0) \mathcal{E}_{n-k}(\sigma_0, \sigma_0) \right) z^n \\
&= 1 + \sum_{k=1}^{\infty} \mathcal{F}_k(\sigma_0, \sigma_0) z^k \left(\sum_{n=k}^{\infty} \mathcal{E}_{n-k}(\sigma_0, \sigma_0) z^{n-k} \right) = 1 + \left(\sum_{k=1}^{\infty} \mathcal{F}_k(\sigma_0, \sigma_0) z^k \right) \cdot \mathcal{G}(z) \\
&= 1 + \mathfrak{F}(z) \mathcal{G}(z).
\end{aligned}$$

□

4 Arboreal complexes

The goal of this Section is study ESBW on arboreal complexes. The two main results are Theorem 1.1, which is proved in Subsections 4.1 and 4.2, and Corollary 1.2, which is proved in Subsection 4.3. The proof of Theorem 1.1 is separated into two parts. First, using the transitive and tree like structure of the regular arboreal complex, we find an explicit formula for $\mathcal{G}(z) = \sum_{n=0}^{\infty} \mathcal{E}_n(\sigma, \sigma) z^n$. Secondly, using the precise expression for $\mathcal{G}(z)$ and the Stieltjes transform the spectral measure is obtained. The proof is similar in spirit to Kesten's proof for k -regular trees [Kes59], however a special care is needed since the terms $\mathcal{E}_n(\sigma, \sigma)$ and $\mathcal{F}_n(\sigma, \sigma)$ are not non-negative as in the graph case.

We start by recalling the definition of arboreal complexes:

Definition 4.1 (Arboreal complexes [PR12, Definition 3.2]). We say that a d -complex is *arboreal* if it is $(d-1)$ -connected, and has no simple d -loops. That is, there are no non-backtracking closed loops of d -cells, $\tau_0, \tau_1, \dots, \tau_n = \tau_0$ such that $\dim(\tau_i \cap \tau_{i+1}) = d-1$ and $\tau_i \neq \tau_{i+2}$ (the chain is non-backtracking). As in the graph case for every $k \geq 1$ there exists a unique k -regular arboreal d -complex denoted T_k^d .

The following choice of oriented cells in T_k^d will be useful for the proof: Choose an arbitrary $(d-1)$ -cell $\sigma_0 \in X_{\pm}^{d-1}$ and call it the 0^{th} layer of T_k^d . Define the 1^{st} layer to be all the oriented $(d-1)$ -cells which are neighbors of σ_0 (there are $k \cdot d$ such cells) and denote one of them by σ_1 . The 2^{nd} layer of T_k^d is the set of oriented $(d-1)$ -cells which are neighbors of a $(d-1)$ -cell in the 1^{st} layer, such that none of their oriented versions are in the 0^{th} or 1^{st} layers. Finally, let σ_2 be a representative in the 2^{nd} layer which is a neighbor of σ_1 . One can continue in the same manner, defining all the layers of T_k^d around σ_0 , eventually ending up with a choice of orientation for T_k^d . Here however, we don't need the full layer structure. Figure 4.1 demonstrates a choice for $\sigma_0, \sigma_1, \sigma_2$ and the layers structure in T_2^2 .

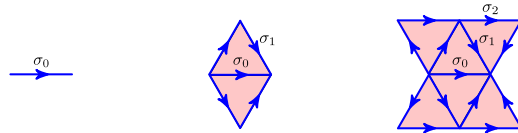


Figure 4.1: The 0^{th} , 1^{st} and 2^{nd} layers in T_2^2 with a choice for σ_0, σ_1 and σ_2 .

4.1 Finding $\mathcal{G}(z)$

Let

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} \mathcal{E}_n^p(\sigma_0, \sigma_0) z^n, \quad \mathfrak{F}(z) = \sum_{n=0}^{\infty} \mathcal{F}_n^p(\sigma_0, \sigma_0) z^n, \quad \mathcal{U}(z) = \sum_{n=0}^{\infty} \mathcal{F}_n^p(\sigma_1, \sigma_0) z^n,$$

and set \mathbf{r} to be the minimum of the radii of convergence of the above power series³.

³As noted in Remark 3.7, $\mathbf{r} \geq \frac{1}{d} > 0$.

Lemma 4.2. For the p -lazy ESBW on T_k^d the following relations hold for every $z \in \mathbb{C}$ such that $|z| < \mathbf{r}$.

$$(1) \mathfrak{F}(z) = pz + (1-p) dz \cdot \mathcal{U}(z).$$

$$(2) \mathcal{U}(z) = pz\mathcal{U}(z) + (1-p) \left[\frac{1}{k}z - \frac{d-1}{k}z\mathcal{U}(z) + \frac{k-1}{k}dz(\mathcal{U}(z))^2 \right].$$

Proof.

(1) By Lemma 3.8(2)

$$\begin{aligned} \mathfrak{F}(z) &= \sum_{n=0}^{\infty} \mathcal{F}_n(\sigma_0, \sigma_0) z^n = \mathcal{F}_1(\sigma_0, \sigma_0) z + \sum_{n=2}^{\infty} \mathcal{F}_n(\sigma_0, \sigma_0) z^n \\ &= pz + \sum_{\sigma'' \in (X^{d-1} \setminus \sigma_0)_{\pm}} E^{\sigma_0} [N_1(\sigma'')] z \cdot \left(\sum_{n=2}^{\infty} \mathcal{F}_{n-1}(\sigma'', \sigma_0) z^{n-1} \right). \end{aligned} \quad (4.1)$$

Due to the tree like structure of T_k^d , for $\sigma'' \in (X^{d-1} \setminus \sigma_0)_{\pm}$ we have $E^{\sigma_0} [N_1(\sigma'')] = 0$ unless σ'' is in the 1st layer of T_k^d . In addition by the transitive structure of T_k^d the power series $\sum_{n=2}^{\infty} \mathcal{F}_{n-1}(\sigma'', \sigma_0) z^{n-1}$ is the same for every σ'' in the first layer of T_k^d and equals $\mathcal{U}(z)$. Thus

$$\mathfrak{F}(z) = pz + \sum_{\substack{\sigma'' \in \text{first} \\ \text{layer of } X_+^{d-1}}} E^{\sigma_0} [N_1(\sigma'')] z \cdot \mathcal{U}(z) = pz + (1-p) z \cdot \mathcal{U}(z).$$

(2) As in part (1) the claim follows by a one step analysis of the ESBW. Using the Markov property, Lemma 3.8(2) and a similar argument to the one in (4.1)

$$\mathcal{U}(z) = \mathcal{F}_1(\sigma_1, \sigma_0) z + \sum_{\sigma'' \in X_{\pm}^{d-1}} E^{\sigma_1} [N_1(\sigma'')] z \cdot \left(\sum_{n=2}^{\infty} \mathcal{F}_{n-1}(\sigma'', \sigma_0) z^{n-1} \right).$$

Due to the tree structure of T_k^d

$$E^{\sigma_1} [N_1(\sigma'')] = \begin{cases} p & , \sigma'' = \sigma_1 \\ (1-p) \frac{1}{k} & , \sigma'' = \sigma_0 \\ (1-p) \frac{1}{k} & , \sigma'' \text{ is in the } 2^{\text{nd}} \text{ layer of } T_k^d \text{ and } \sigma'' \sim \sigma_1 \\ (1-p) \frac{1}{k} & , \overline{\sigma''} \text{ is in the } 1^{\text{st}} \text{ layer of } T_k^d \text{ and } \sigma'' \cup \sigma_1 \text{ is a } d\text{-cell} \\ 0 & , \text{otherwise} \end{cases}$$

and by its transience

$$\sum_{n=2}^{\infty} \mathcal{F}_{n-1}(\sigma'', \sigma_0) z^{n-1} = \begin{cases} \mathcal{U}(z) & , \sigma'' = \sigma_1 \\ \sum_{n=1}^{\infty} \mathcal{F}_n(\sigma_2, \sigma_0) z^n & , \sigma'' \text{ is in the } 2^{\text{nd}} \text{ layer of } T_k^d \text{ and } \sigma'' \sim \sigma_1 \\ -\mathcal{U}(z) & , \overline{\sigma''} \text{ is in the } 1^{\text{st}} \text{ layer of } T_k^d \text{ and } \sigma'' \cup \sigma_1 \text{ is a } d\text{-cell} \end{cases}.$$

Finally, note that the number of σ'' in the 2nd layer of T_k^d such that $\sigma'' \sim \sigma_1$ is exactly $d(k-1)$ and that the number of σ'' such that $\overline{\sigma''}$ is in the 1st layer and $\sigma'' \cup \sigma_1$ is a d -cell is exactly $d-1$. Combining all of the above gives

$$\mathcal{U}(z) = pz \cdot \mathcal{U}(z) + (1-p) \left[\frac{1}{k}z - \frac{d-1}{k}z \cdot \mathcal{U}(z) + \frac{k-1}{k}dz \cdot \sum_{n=1}^{\infty} \mathcal{F}_n(\sigma_2, \sigma_0) z^n \right].$$

Thus, the proof will be complete once we show that $\sum_{n=1}^{\infty} \mathcal{F}(\sigma_2, \sigma_0) z^n = (\mathcal{U}(z))^2$. Since each particle starting in σ_2 must split through either σ_1 or $\overline{\sigma_1}$ in order to reach σ_0 we can rewrite $\mathcal{F}_n(\sigma_2, \sigma_0)$ as a sum according to the first “visit” to one of these cells. This gives

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathcal{F}_n(\sigma_2, \sigma_0) z^n &= \sum_{n=0}^{\infty} E^{\sigma_2} [F_n(\sigma_0)] z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n [E^{\sigma_2} [K_k(\sigma_1)] E^{\sigma_1} [F_{n-k}(\sigma_0)] + E^{\sigma_2} [K_k(\overline{\sigma_1})] E^{\overline{\sigma_1}} [F_{n-k}(\sigma_0)]] z^n \\
&\stackrel{(*)}{=} \sum_{n=0}^{\infty} \sum_{k=0}^n [E^{\sigma_2} [K_k(\sigma_1)] E^{\sigma_1} [F_{n-k}(\sigma_0)] - E^{\sigma_2} [K_k(\overline{\sigma_1})] E^{\sigma_1} [F_{n-k}(\sigma_0)]] z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n E^{\sigma_2} [F_k(\sigma_1)] E^{\sigma_1} [F_{n-k}(\sigma_2)] z^n = \sum_{n=0}^{\infty} \sum_{k=0}^n \mathcal{F}_k(\sigma_2, \sigma_1) z^k \cdot \mathcal{F}_{n-k}(\sigma_1, \sigma_0) z^{n-k} \\
&= (\mathcal{U}(z))^2,
\end{aligned}$$

where for $(*)$ we used the fact that $E^{\overline{\sigma}} [K_k(\sigma')] = E^{\sigma} [K_k(\overline{\sigma'})]$ for every $k \geq 0$ and $\sigma, \sigma' \in (T_k^d)_{\pm}^{d-1}$. \square

Using Lemma 4.2 we can now find $\mathcal{G}(z)$. For simplicity fix $p = 0$ and note that in this case Lemma 4.2(2) gives $\mathcal{U}(z) = \frac{1}{k}z - \frac{d-1}{k}z\mathcal{U}(z) + \frac{k-1}{k}dz(\mathcal{U}(z))^2$. The solutions of the equation are

$$L_{\pm}(z) = \frac{(d-1)z + k \pm \sqrt{((d-1)z + k)^2 - 4(k-1)dz^2}}{2(k-1)dz},$$

and since only the solution L_- satisfies $L_-(0) = 0 = \mathcal{U}(0)$ we conclude that $\mathcal{U} = L_-$. Using Lemma 4.2(1) and Proposition 3.6 it follows that as long as $\mathfrak{F}(z) \neq 1$

$$\mathcal{G}(z) = \frac{1}{1 - \mathfrak{F}(z)} = \frac{1}{1 - dz\mathcal{U}(z)}$$

which gives

$$\mathcal{G}(z) = \frac{2(k-1)}{k - 2 - (d-1)z + \sqrt{((d-1)z + k)^2 - 4(k-1)dz^2}}.$$

Note that the singularity points of \mathcal{G} are the points where the denominator is zero (which are in fact the points where $\mathfrak{F}(z) = 1$) and the points where the square-root is zero. Those are given by

$$z = 1, \text{ when } k \leq d+1$$

and

$$z_{\pm} = \frac{k}{1 - d \mp 2\sqrt{(k-1)d}}.$$

In particular we infer that

$$\left(\limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{E}_n^0(\sigma_0, \sigma_0)|} \right)^{-1} = \left(\text{radius of convergence of } \mathcal{G} \right) = \begin{cases} \min \left\{ 1, \left| \frac{k}{d-1+2\sqrt{(k-1)d}} \right| \right\} & k \leq d+1 \\ \frac{k}{d-1+2\sqrt{(k-1)d}} & k > d+1 \end{cases}.$$

4.2 Finding the spectral measure

Once the moment generating function $\mathcal{G}(z)$ is known the spectral measure can be calculated using the Stieltjes transform. Let $\mu_{d,k}$ be the spectral measure associated with the operator \mathcal{A}_0 of the arboreal complex T_k^d and for $z \in \mathbb{C} \setminus \mathbb{R}$ let $\mathcal{S}(z) = \int_{\mathbb{R}} \frac{1}{x-z} d\mu_{d,k}(x)$ be its Stieltjes transform. Note that for $z \in \mathbb{C} \setminus \mathbb{R}$ whose absolute value is bigger than $\max\{|\lambda| : \lambda \in \text{support of } \mu_{d,k}\}$

$$\begin{aligned} \mathcal{S}(z) &= \int_{\mathbb{R}} \frac{1}{x-z} d\mu_{d,k}(x) = -\frac{1}{z} \int_{\mathbb{R}} \frac{1}{1-\frac{x}{z}} d\mu_{d,k}(x) \\ &= -\frac{1}{z} \int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\frac{x}{z}\right)^n d\mu_{d,k}(x) = -\frac{1}{z} \sum_{n=0}^{\infty} \mathcal{E}_n(\sigma_0, \sigma_0) \frac{1}{z^n} = -\frac{1}{z} \mathcal{G}\left(\frac{1}{z}\right). \end{aligned}$$

Since $\mathcal{S}(z)$ and $-\frac{1}{z} \mathcal{G}\left(\frac{1}{z}\right)$ agree on an open ball it follows that their analytic continuations⁴ agree and in particular that

$$\mathcal{S}(z) = -\frac{2(k-1)}{(k-2)z - (d-1) + \sqrt{(d-1+kz)^2 - 4(k-1)d}}.$$

Having found the Stieltjes transform \mathcal{S} we turn to evaluate $\mu_{d,k}$ starting with the spectral density $\rho_{d,k}$, namely, the Radon-Nikodym derivative of the absolutely continuous part. This is done by evaluating the limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varepsilon}{(x-x_0)^2 + \varepsilon^2} d\mu_{d,k}(x)$$

which by the dominated convergence theorem equals $\rho_{d,k}(x_0)$ when $\mu_{d,k}$ doesn't have an atom in x_0 and $+\infty$ when it does. For every $x_0 \in \mathbb{R}$ (except for $x_0 = 1$ when $k \leq d+1$)

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varepsilon}{(x-x_0)^2 + \varepsilon^2} d\mu_{d,k}(x) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \text{Im}(\mathcal{S}(x_0 + i\varepsilon)) \\ &= -\frac{1}{\pi} \text{Im} \left(\frac{2(k-1)}{(k-2)x_0 - (d-1) + \sqrt{(d-1+kx_0)^2 - 4(k-1)d}} \right). \end{aligned} \tag{4.2}$$

The right hand side of (4.2) equals zero whenever $(d-1+kx_0)^2 \geq 4(k-1)d$ and

$$\frac{\sqrt{4(k-1)d - (d-1+kx)^2}}{2\pi(d+x)(1-x)}$$

when $(d-1+kx_0)^2 \leq 4(k-1)d$. Since $(d-1+kx_0)^2 \leq 4(k-1)d$ exactly when

$$x_0 \in I_{d,k} \equiv \left[\frac{1-d-2\sqrt{(k-1)d}}{k}, \frac{1-d+2\sqrt{(k-1)d}}{k} \right],$$

it follows that the density function is

$$\rho_{d,k}(x) = \frac{\sqrt{4(k-1)d - (d-1+kx)^2}}{2\pi(d+x)(1-x)} \chi_{x \in I_{d,k}}.$$

⁴One can avoid the use of analytic continuation by working with the p -lazy SBRW for any $p > \frac{d-1}{d+1}$.

One can now verify that

$$\int_{I_{d,k}} \rho_{d,k}(x) dx = \begin{cases} \frac{k}{d+1} & , k < d+1 \\ 1 & , k \geq d+1 \end{cases},$$

which suggest that the size of the atom in the unique suspect for being one, i.e., $x_0 = 1$ when $k \leq d+1$, is $\frac{d+1-k}{d+1}$. A direct proof of this fact without calculating the above integral can also be given using \mathcal{S} . Define

$$h(\varepsilon) := -i \int_{\mathbb{R}} \frac{\varepsilon}{x-1-i\varepsilon} d\mu_{d,k}(x) = -i\varepsilon \mathcal{S}(1+i\varepsilon).$$

By the dominated convergence theorem $\lim_{\varepsilon \downarrow 0} h(\varepsilon) = \mu_{d,k}(\{1\})$ and therefore

$$\mu_{d,k}(\{1\}) = \lim_{\varepsilon \downarrow 0} -i\varepsilon \mathcal{S}(1+i\varepsilon) = \begin{cases} \frac{d+1-k}{d+1} & , k \leq d+1 \\ 0 & , k = d+1 \end{cases}.$$

This shows that

$$\mu_{d,k}(A) = \begin{cases} \int_A \rho_{d,k}(x) dx + \frac{d+1-k}{d+1} \chi_{0 \in A} & , k < d+1 \\ \int_A \rho_{d,k}(x) dx & , k \geq d+1 \end{cases}$$

and completes the proof. \square

Let us take this opportunity to state a conjecture regarding the eigenvalue 1 in simplicial complexes. Theorem 1.1 implies in particular that 1 is an eigenvalue of \mathcal{A}_0 in T_k^d as long as $k \leq d$. We conjecture that this holds in a much bigger generality:

Conjecture 4.3. *One is an eigenvalue of \mathcal{A}_0 for every d -complex X such that $\sup_{\sigma \in X^{d-1}} \deg(\sigma) \leq d$.*

A weak version of the conjecture is:

Conjecture 4.4 (Weaker version). *One is an eigenvalue of \mathcal{A}_0 for every arboreal d -complex X such that $\sup_{\sigma \in X^{d-1}} \deg(\sigma) \leq d$.*

4.3 Transience and recurrence of ESBW on regular arboreal complexes

The notion of transient $(d-1)$ -walk was defined in [PR12, Subsection 3.8]. A slightly more general analogue for the ESBW is:

Definition 4.5. The ESBW is called transient if $\sum_{n=0}^{\infty} \mathcal{E}_n^p(\sigma, \sigma) < \infty$ for every $\sigma \in X^{d-1}$ and some $\frac{d-1}{d+1} < p < 1$. If $\sum_{n=0}^{\infty} \mathcal{E}_n^p(\sigma, \sigma) < \infty$ for some $\sigma \in X^{d-1}$ and $\frac{d-1}{d+1} < p < 1$ the random walk is called recurrent.

Let X be a d -complex and denote by μ^p the spectral measure of \mathcal{A}_p associated with the function $\mathbb{1}_{\sigma}$. Since $\mathcal{A}_p = pI + (1-p)\mathcal{A}_0$, it follows that $\int_{\mathbb{R}} f(x) d\mu^p(x) = \int_{\mathbb{R}} f(p + (1-p)x) d\mu^0(x)$ for every integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$. In addition, since $\text{Support}(\mu^p) \subset \text{Spec}(\mathcal{A}_p) \subset [1 - (1-p)(d+1), 1]$, it follows that the support of the measure μ^p is contained in $(-1, 1]$ for every $\frac{d-1}{d+1} < p < 1$. Therefore, by the monotone convergence theorem and the relation between μ^0 and μ^p

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_n^p(\sigma, \sigma) &= \sum_{n=0}^{\infty} \deg(\sigma) \cdot \langle \mathcal{A}_p^n \mathbb{1}_{\sigma}, \mathbb{1}_{\sigma} \rangle = \deg(\sigma) \cdot \sum_{n=0}^{\infty} \int_{\mathbb{R}} x^n d\mu^p(x) \\ &= \deg(\sigma) \cdot \int_{\mathbb{R}} \frac{1}{1-x} d\mu^p(x) = \deg(\sigma) \cdot \int_{\mathbb{R}} \frac{1}{1-(p+(1-p)x)} d\mu^0(x) \\ &= \frac{1}{1-p} \cdot \int_{\mathbb{R}} \frac{1}{1-x} d\mu^0(x). \end{aligned} \tag{4.3}$$

In particular the p -lazy branching random walk is recurrent/transient for some $\frac{d-1}{d+1} < p < 1$ if and only if it is recurrent/transient for every such p .

Proof of Corollary 1.2. By the above argument, in order to check recurrence/transience of the ESRW it suffices to check whether the integral $\int_{\mathbb{R}} \frac{1}{1-x} d\mu_{d,k}(x)$ is infinite/finite respectively. When $k \leq d$, 1 is an atom of the measure $\mu_{d,k}$ and therefore the integral is infinite. If $k > d + 1$ the spectrum of $\mu_{d,k}$ is a compact subset of $(-\infty, 1)$ and therefore the integral is finite. Finally, in the case $k = d + 1$

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{1-x} d\mu_{d,k}(x) &= \int_{I_{d,k}} \frac{1}{1-x} \cdot \frac{\sqrt{4d^2 - ((d+1)x + (d-1))^2}}{2\pi(d+x)(1-x)} dx \\ &= \int_{\frac{1-3d}{d+1}}^1 \frac{\sqrt{(d+1)((d+1)x + (3d-1))}}{2\pi(d+x)} \frac{1}{(1-x)^{\frac{3}{2}}} dx = \infty, \end{aligned}$$

which implies that the ESRW on T_{d+1}^d is recurrent. \square

5 Dirichlet problem on simplicial complexes

Dirichlet problem concerns with finding a function that solves a partial differential equation (PDE) with prescribed boundary values. The PDE which is usually under consideration is Laplace's equation.

In the discrete setting of graphs Dirichlet problem is stated as follows:

Discrete Dirichlet problem: Given a finite graph $G = (V, E)$, a non-empty subset $A \subset V$ and a function $f : A \rightarrow \mathbb{R}$ find a solution $F : V \rightarrow \mathbb{R}$ to the boundary value problem

$$\begin{cases} \Delta^+ F(x) & , \forall x \in V \setminus A \\ F(x) = f(x) & , \forall x \in A \end{cases}.$$

If G is a connected graph, then for every non-empty set $A \subset V$ and $f : A \rightarrow \mathbb{R}$ there exists a solution given by $F(x) = E^x[f(Y_{\tau_A})]$, where $(Y_n)_{n \geq 0}$ is the simple random walk on the graph G and $\tau_A = \inf \{k \geq 0 : Y_k \in A\}$. In addition, the solution is unique due to the maximum principle.

A high-dimensional counterpart of the problem for forms is:

High-dimensional discrete Dirichlet problem: Given a finite d -complex X , a non-empty subset $A \subset X^{d-1}$ and a form $f : A_{\pm} \rightarrow \mathbb{R}$ (where A_{\pm} is the set of oriented $(d-1)$ -cells whose unoriented version is in A) find a solution $F \in \Omega^{d-1}$ to the boundary value problem

$$\begin{cases} \Delta^+ F(\sigma) & , \forall \sigma \in (X^{d-1} \setminus A)_{\pm} \\ F(\sigma) = f(\sigma) & , \forall \sigma \in A_{\pm} \end{cases}.$$

The situation in high dimensions is more involved and for a general set A one can have infinitely many solutions. For example if X is composed of a single triangle $t = \{v_0, v_1, v_2\}$, $A = \{\{v_0, v_1\}\}$ and $f(\{v_0, v_1\}) = -f(\{v_1, v_0\}) = 1$, the form defined by

$$F_{\alpha}(e) = \begin{cases} 1 & , e = [v_0, v_1] \\ \alpha & , e = [v_1, v_2] \\ -1 - \alpha & , e = [v_2, v_0] \end{cases}$$

is a solution to the Dirichlet problem for every $\alpha \in \mathbb{R}$.

Before turning to discuss the existence and uniqueness of solutions to the high-dimensional Dirichlet problem some additional definitions are required. Let X be a d -complex and $\emptyset \neq A \subset X^{d-1}$. Since the case $A = X^{d-1}$ is degenerate and has exactly one solution, $F = f$, we assume without loss of generality that $A \neq X^{d-1}$.

Consider Δ^+ as a matrix and denote by $\Delta_{X \setminus A}^+$ its restriction to rows and columns of $(d-1)$ -cells in $X^{d-1} \setminus A$. Similarly let $\delta_d^{X \setminus A}$ be the restriction of δ_d to $(d-1)$ -cells in $X^{d-1} \setminus A$.

Define the A -absorbing, p -lazy SBRW on X to be the usual SBRW except that any particle in A_\pm stays put with probability one. Let P_A, E_A denote the probability and expectation of the A -absorbing SBRW respectively. We can now define the related effective process $(D_n)_{n \geq 0}$ and its Green function

$$\mathcal{G}_A^p(\sigma, \sigma') = \sum_{n=0}^{\infty} E_A^\sigma[D_n(\sigma)], \quad \forall \sigma, \sigma' \in (X^{d-1} \setminus A)_\pm.$$

Our goal is to prove the following Theorem:

Theorem 5.1 (Solution to the high-dimensional Dirichlet problem). *Let X be a finite d -complex, $\frac{d-1}{d+1} < p < 1$, $\emptyset \neq A \subsetneq X^{d-1}$ such that $\Delta_{X \setminus A}^+$ is invertible and $f : A_\pm \rightarrow \mathbb{R}$. Then the unique solution to the Dirichlet problem related to the triplet (X, A, f) is the function $F : X_\pm^{d-1} \rightarrow \mathbb{R}$ given by*

$$F(\sigma) = \frac{1}{1-p} \sum_{\sigma'' \in A_\pm} \left(\sum_{\substack{\sigma' \in (X^{d-1} \setminus A)_\pm \\ \sigma' \sim \sigma''}} \frac{\mathcal{G}_A^p(\sigma, \sigma')}{\deg(\sigma')} \right) f(\sigma'').$$

Proof. Decompose⁵ the matrix representation of Δ^+ as $\Delta^+ = \left(\begin{array}{c|c} \Delta_A^+ & -R \\ \hline -Q & \Delta_{X \setminus A}^+ \end{array} \right)$. Then F is a solution to the Dirichlet problem if and only if

$$\left(\begin{array}{c|c} I & 0 \\ \hline -Q & \Delta_{X \setminus A}^+ \end{array} \right) \begin{pmatrix} \vdots \\ F \\ \vdots \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Note that this operator is invertible if and only if the operator $\Delta_{X \setminus A}^+$ is invertible, and since the invertibility of $\Delta_{X \setminus A}^+$ was assumed it follows that there exists a unique solution to Dirichlet problem given by $F|_{X^{d-1} \setminus A} = \left(\Delta_{X \setminus A}^+ \right)^{-1} Qf$. Next we show that whenever $\Delta_{X \setminus A}^+$ is invertible and $\frac{d-1}{d+1} < p < 1$ its inverse is given by $\frac{1}{1-p} \mathcal{G}_A^p$ and in particular that \mathcal{G}_A^p is well defined. Indeed, by the same argument as in [PR12, Proposition 2.7(2)] the spectrum of $\Delta_{X \setminus A}^+$ is always a subset of $[0, d+1]$ and due to the fact that $\Delta_{X \setminus A}^+$ is invertible $\text{Spec}(\Delta_{X \setminus A}^+) \subset [\lambda_0, d+1]$ for some $\lambda_0 > 0$. Defining the operator $\mathcal{A}_p^{X \setminus A} := I - (1-p) \Delta_{X \setminus A}^+$, it follows that $\text{Spec}(\mathcal{A}_p^{X \setminus A}) \subset [1 - (1-p)(d+1), 1 - (1-p)\lambda_0]$. Since $\frac{d-1}{d+1} < p < 1$ this is a closed sub-interval of $(-1, 1)$ and so $\|\mathcal{A}_p^{X \setminus A}\| = \sup\{|\lambda| : \lambda \in \text{Spec}(\mathcal{A}_p)\} < 1$. Noting that $\mathcal{A}_p^{X \setminus A}$ is the “transition” operator of the A -absorbing ESBRW for $(d-1)$ -cells in $X^{d-1} \setminus A$ it follows that $E_A^\sigma[D_n(\sigma')] = \deg(\sigma') \cdot \left\langle \left(\mathcal{A}_p^{X \setminus A} \right)^n \mathbb{1}_\sigma, \mathbb{1}_{\sigma'} \right\rangle$ for every $\sigma, \sigma' \in (X^{d-1} \setminus A)_\pm$. Thus we conclude that

$$\begin{aligned} |\mathcal{G}_A^p(\sigma, \sigma')| &= \left| \sum_{n=0}^{\infty} E_A^\sigma[D_n(\sigma')] \right| = \deg(\sigma') \left| \sum_{n=0}^{\infty} \left\langle \left(\mathcal{A}_p^{X \setminus A} \right)^n \mathbb{1}_\sigma, \mathbb{1}_{\sigma'} \right\rangle \right| \\ &\leq \deg(\sigma') \sum_{n=0}^{\infty} \|\mathcal{A}_p^{X \setminus A}\|^n < \infty, \quad \forall \sigma, \sigma' \in (X^{d-1} \setminus A)_\pm, \end{aligned}$$

⁵The decomposition here is according to whether the $(d-1)$ -cells are in A or in $(X^{d-1} \setminus A)$.

which in particular shows that \mathcal{G}_A^p is well defined.

The fact that $\frac{1}{1-p}\mathcal{G}_A^p = \left(\Delta_{X \setminus A}^+\right)^{-1}$ follows now from the Markov property. Indeed,

$$\begin{aligned} \mathcal{G}_A^p(\sigma, \sigma') &= \sum_{n=0}^{\infty} E_A^\sigma[D_n(\sigma')] = \mathbb{1}_\sigma(\sigma') + \sum_{n=1}^{\infty} \sum_{\sigma'' \in (X^{d-1} \setminus A)_\pm} E_A^\sigma[N_1(\sigma'')] E_A^{\sigma''}[D_{n-1}(\sigma')] \\ &= \mathbb{1}_\sigma(\sigma') + \left(\mathcal{A}_p^{X \setminus A} \mathcal{G}_A^p\right)(\sigma, \sigma') \end{aligned}$$

which gives $\frac{1}{1-p}\mathcal{G}_A^p \Delta_{X \setminus A}^+ = \mathcal{G}_A^p (I - \mathcal{A}_p^{X \setminus A}) = I$.

Finally, note that Q is nothing else than the restriction of $-\Delta^+ = \frac{1}{1-p}(\mathcal{A}_p - I)$ to columns of $(d-1)$ -cells in A and rows of $(d-1)$ -cells in $X^{d-1} \setminus A$. Since there are no diagonal elements in the restriction this is the same as the restriction of $\frac{1}{1-p}\mathcal{A}_p$ to the same rows and columns which for $\sigma' \in (X^{d-1} \setminus A)_\pm$ and $\sigma'' \in A_\pm$ equals $\frac{1}{1-p}E^{\sigma'}[N_1(\sigma'')] = \begin{cases} \frac{1}{\deg(\sigma')} & , \sigma'' \sim \sigma' \\ 0 & , \text{otherwise} \end{cases}$. \square

Before turning to the next section we wish to discuss the main condition in Theorem 5.1, namely, the invertibility of $\Delta_{X \setminus A}^+$. We start with some simple observations:

Claim 5.2 (Invertibility of $\Delta_{X \setminus A}^+$). Let X be a finite d -complex and $\emptyset \neq A \subsetneq X^{d-1}$. The following are equivalent:

- (1) $\Delta_{X \setminus A}^+$ is invertible.
- (2) $\ker \delta_d^{X \setminus A} = \ker \Delta_{X \setminus A}^+$ is trivial.
- (3) For every form $f : (X^{d-1} \setminus A)_\pm \rightarrow \mathbb{R}$ which is not identically zero, the extension $\tilde{f} : X_\pm^{d-1} \rightarrow \mathbb{R}$ given by $\tilde{f}(\sigma) = \begin{cases} f(\sigma) & \sigma \in (X^{d-1} \setminus A)_\pm \text{ is not in } \ker \delta_d = \ker \Delta^+ = Z^{d-1}. \\ 0 & \sigma \in A_\pm \end{cases}$
- (4) The relative homology $H_d(X, A)$ (see [Hat02, Section 2.1] for the definition) is trivial.

Using the above equivalent definitions we can identify some cases in which it is easier to check whether $\Delta_{X \setminus A}^+$ is invertible or not. Let us start with two definitions:

Definition 5.3. Let X be a finite d -complex and $\emptyset \neq A \subsetneq X^{d-1}$. The set A is called *exhaustive* for the complex X if there exists a finite sequence $A = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_N = X^{d-1}$ such that for every $n \geq 1$ and $\sigma \in A_n$, one can find $\tau \in \text{cf}(\sigma)$ for which $\text{face}(\tau) \setminus \sigma \subset A_{n-1}$.

Definition 5.4. [DKM09, Definition 3.1] Let X be a d -complex and $k \leq d$. A k -dimensional *simplicial spanning tree* (k -SST for short) of X is a k -dimensional subcomplex $Y \subset X$ such that $Y^{k-1} = X^{k-1}$, $H_k(Y; \mathbb{Z}) = 0$ and $|H_{k-1}(Y; \mathbb{Z})| < \infty$, where $H_l(Y; \mathbb{Z})$ are the homology groups with coefficients in \mathbb{Z} (see [Hat02, Section 2.1] for the definition).

Lemma 5.5. *Let X be a finite d -complex.*

- (1) *When $d = 1$, $\Delta_{X \setminus A}^+$ is invertible if and only if A contains a vertex in each of the 0-components of X .*
- (2) *If A is an exhaustive set for the complex X , then $\Delta_{X \setminus A}^+$ is invertible.*
- (3) *If A is a deformation retract of X and $\Delta_{X \setminus A}^+$ is invertible then $H_d(X) = 0$.*

- (4) If there exists $\varrho \in X^{d-2}$ such that $\text{cf}(\varrho) \subset X^{d-1} \setminus A$, then $\Delta_{X \setminus A}^+$ is not invertible.
- (5) If $|A| = |X^d|$ then $\Delta_{X \setminus A}^+$ is invertible if and only if X is a d -SST of X and $X^{d-2} \cup (X \setminus A)$ is a $(d-1)$ -SST of X .

Proof.

- (1) For every $f \in \Omega^0(X)$

$$\delta_1^{X \setminus A} f = \sum_{x \sim y} (f(x) \chi_{x \notin A} - f(y) \chi_{y \notin A}).$$

Thus $\delta_1^{X \setminus A} f = 0$ implies that f is constant on every connected component and is zero on every component containing a vertex in A .

- (2) Assume that A is exhaustive with an exhausting sequence $(A_n)_{0 \leq n \leq N}$ and that $f \in \ker \Delta_{X \setminus A}^+ = \ker \delta_d^{X \setminus A}$. For every $\sigma \in A_1 \setminus A_0$ one can find $v \triangleleft \sigma$ such that all $(d-1)$ -faces of $v\sigma$ except for σ itself are in A_0 and therefore

$$0 = \delta_d^{X \setminus A} f(v\sigma) = \sum_{i=0}^d f((v\sigma) \setminus (v\sigma)_i) \cdot \chi_{(v\sigma) \setminus (v\sigma)_i \notin A_\pm} = f(\sigma).$$

Consequently $f|_{(A_1 \setminus A_0)_\pm} \equiv 0$. One can now proceed by induction to show that $f|_{(A_n \setminus A_0)_\pm} \equiv 0$ for every $1 \leq n \leq N$. Since the case $n = N$ implies $f|_{(X^{d-1} \setminus A)_\pm} \equiv 0$ the kernel of $\delta_d^{X \setminus A}$ is trivial and the result follows.

- (3) Since $X^{d-2} \cup A$ is a $(d-1)$ -complex, $H_d(X^{d-2} \cup A) = 0$. In addition by Claim 5.2 and the assumption that $\Delta_{X \setminus A}^+$ is invertible we have $H_d(X, A) = 0$. The result now follows since whenever A is a deformation retract of X the sequence

$$0 \rightarrow H_d(X^{d-2} \cup A) \rightarrow H_d(X) \rightarrow H_d(X, A) \rightarrow H_{d-1}(X^{d-2} \cup A) \rightarrow \dots$$

is exact (see [Hat02, Theorem 2.13]).

- (4) If $\varrho \in X^{d-2}$ and $\text{cf}(\varrho) \subset X^{d-1} \setminus A$ then the support of the form $\tilde{f} = \delta_{d-1} \mathbb{1}_\varrho \neq 0$ is contained in $X^{d-1} \setminus A$ (thus making it the extension of $f = \tilde{f}|_{X^{d-1} \setminus A}$). Since $\delta_d \tilde{f} = \delta_d \delta_{d-1} \mathbb{1}_\varrho = 0$, this implies by Claim 5.2 that $\Delta_{X \setminus A}^+$ is not invertible.
- (5) This is the content of [DKM09, Proposition 4.1], see also [Kal83] for a discussion on the complete skeleton case.

□

6 Lower simplicial branching random walk

Let X be a d -complex such that $M = \sup_{\sigma \in X^{d-1}} \deg(\sigma) < \infty$. In [MS13], Mukherjee and Steenbergen defined a version of the random walk on simplicial complexes correlated with the lower Laplacian instead of the upper one. This is done by defining a new neighboring relation, which we call the adjacency relation, that uses faces instead of cofaces, see Definition 2.3. The lower random walk (which is called by the authors the Dirichlet random walk) is then defined as follows:

Definition 6.1 ([MS13, Definition 3.1]). The p -lazy, d -lower random walk is a Markov chain $(Z_n)_{n \geq 0}$ on $X_{\pm}^d \uplus \{\Theta\}$ (where Θ is an additional absorbing state) with transition probabilities

$$\text{Prob} \left(Z_n = \sigma' \middle| Z_n = \sigma \right) = \begin{cases} p & , \sigma' = \sigma, \sigma, \sigma' \neq \Theta \\ \frac{1-p}{(M-1)(d+1)} & , \sigma' \underset{\downarrow}{\sim} \sigma, \sigma, \sigma' \neq \Theta \\ (1-p) \left[1 - \frac{1}{(M-1)(d+1)} \cdot \sum_{\tau \in \text{face}(\sigma)} \deg(\tau) \right] & , \sigma' = \Theta, \sigma \neq \Theta \\ 1 & , \sigma', \sigma = \Theta \\ 0 & , \text{otherwise} \end{cases}.$$

As for the upper walk define the heat kernel $\mathbf{p}_n^\downarrow(\sigma, \sigma') = \text{Prob} \left(Z_n = \sigma' \middle| Z_0 = \sigma \right)$, the lower expectation process $\mathcal{E}_n^\downarrow(\sigma, \sigma') = \mathbf{p}_n^\downarrow(\sigma, \sigma') - \mathbf{p}_n^\downarrow(\sigma, \overline{\sigma'})$ and its normalized version

$$\tilde{\mathcal{E}}_n^\downarrow(\sigma, \sigma') = \left(\frac{M-1}{p(M-2)+1} \right)^n \mathcal{E}_n^\downarrow(\sigma, \sigma').$$

The following proposition summarizes some of the results proved in [MS13] regarding the connection between the d -lower random walk and the d -homology of the complex:

Proposition 6.2 ([MS13, Proposition 1.1]). *Let X be a finite d -complex such that $M = \sup_{\sigma \in X^{d-1}} \deg(\sigma) < \infty$ and $\tilde{\Delta}_d^-$ the d -lower Laplacian⁶ given by $\tilde{\Delta}_d^- f(\sigma) = (d+1)f(\sigma) - \sum_{\sigma' \underset{\downarrow}{\sim} \sigma} f(\sigma')$.*

(1) *The time evolution of $\tilde{\mathcal{E}}_n^\downarrow(\sigma, \sigma')$ is given by $\tilde{\mathcal{E}}_n^\downarrow(\cdot, \sigma') = \left(B_p \tilde{\mathcal{E}}_{n-1}^\downarrow \right)(\cdot, \sigma')$ where B_p acts on the first coordinate and is given by*

$$B_p = \frac{p(M-2)+1}{M-1} I - \frac{1-p}{(M-1)(d+1)} \tilde{\Delta}_d^-.$$

(2) *If $\frac{M-2}{3M-4} < p < 1$, then $\tilde{\mathcal{E}}_\infty^\downarrow = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n^\downarrow$ always exists.*

(3) *If $\frac{M-2}{3M-4} < p < 1$, then $\left\{ \tilde{\mathcal{E}}_\infty^\downarrow(\sigma, \cdot) \right\}_{\sigma \in X_{\pm}^{d-1}} \subset B^{d-1}$ if and only if $H_d(X) = 0$.*

(4) *If furthermore $p \geq \frac{1}{2}$ then*

$$\text{dist} \left(\tilde{\mathcal{E}}_n^\downarrow, \tilde{\mathcal{E}}_\infty^\downarrow \right) = O \left(\left(1 - \frac{1-p}{(p(M-2)+1)(d+1)} \tilde{\lambda}^\downarrow(X) \right)^n \right),$$

$$\text{where } \tilde{\lambda}^\downarrow(X) = \min \left(\text{Spec} \left(\tilde{\Delta}_d^- \big|_{(B^{d+1})^\perp} \right) \right) = \min \left(\text{Spec} \left(\tilde{\Delta}_d^- \right) \right).$$

Remark 6.3. One can also use the lower expectation process in order to find the dimension of $H_d(X)$ as in Theorem 2.10 and Theorem 3.1.

As in the case of the upper Laplacian we define a new stochastic process, called the *lower simplicial branching random walk*, LSBRW for short, which is connected to the spectrum of the lower Laplacian in a similar way as the d -lower random walk. The effective process generated from the LSBRW has the property of being a sequence of random forms in $\Omega^d(X)$ already in the process level. In addition it doesn't require normalization and there is no need for the additional absorbing state.

The p -lazy lower simplicial branching random walk on X is a time-homogeneous Markov chain $\left(N_n^\downarrow(\cdot) \right)_{n \geq 0}$ with state space $\mathbb{N}^{X_{\pm}^d}$ which counts the number of particles at time n on any of the oriented d -cells, that is:

⁶This specific lower Laplacian is related to the weight function $w \equiv 1$.

- N_n^\downarrow is a random function from X_\pm^d to \mathbb{N} .
- The process is Markovian, i.e., $\text{Prob}\left(N_n^\downarrow \in A \mid N_1^\downarrow, \dots, N_{n-1}^\downarrow\right) = \text{Prob}\left(N_n^\downarrow \in A \mid N_{n-1}^\downarrow\right)$ and time homogeneous, namely $\text{Prob}\left(N_n^\downarrow = g \mid N_{n-1}^\downarrow = f\right)$ doesn't depend on n .
- $N_n^\downarrow(\sigma)$ is the random number of particles in τ at time n for every $\tau \in X_\pm^d$ and $n \geq 0$.

One step evolution of the process (its transition kernel) is defined as follows: Given a configuration of particles on X_\pm^d all the particles evolve simultaneously and independently. If a particle is positioned in τ , then it stays put with probability p , and with probability $1 - p$ chooses one of the faces of τ uniformly at random and splits into new particles which are now positioned on the d -cells adjacent to τ whose intersection with τ is the chosen face (one particle on each such d -cell). Note that one step of the process is comprised of the evolution of all existing particles. An illustration of one step of the lower simplicial branching random walk on a triangle complex can be found in Figure 6.1.

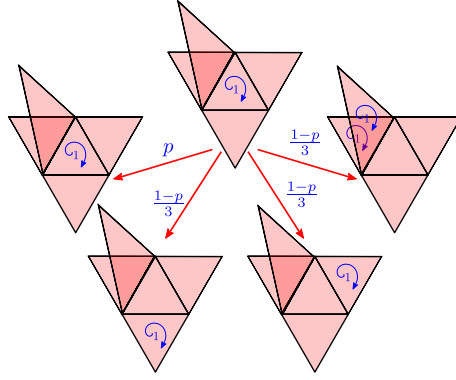


Figure 6.1: One step of the LSBRW on a triangle complex.

The effective LSBRW is now defined by

$$D_n^\downarrow(\sigma) = N_n^\downarrow(\sigma) - N_n^\downarrow(\bar{\sigma}),$$

and its heat kernel is

$$\mathcal{E}_n^\downarrow(\sigma, \sigma') = E^\sigma[D_n^\downarrow(\sigma')].$$

As before, the notation $\mathcal{E}_n^{\downarrow,p}(\sigma, \sigma')$ is used to stress the dependence on p .

Remark 6.4. Similarly to the case of SBRW, one can consider several variants of the process. One can also define the model on k -cells in a d -complex for every $1 \leq k \leq d$. This however is equivalent to studying the process on X^k instead of X and thus falls back to the above setting.

A similar argument to the one in Lemma 3.2(1) shows that for $n \geq 1$

$$\mathcal{E}_n^\downarrow(\sigma, \sigma') = p\mathcal{E}_{n-1}^\downarrow(\sigma, \sigma') + \frac{1-p}{d+1} \sum_{\sigma'' \sim \sigma} \mathcal{E}_{n-1}^\downarrow(\sigma'', \sigma') = (I - (1-p)\Delta_d^-) \mathcal{E}_{n-1}^\downarrow(\sigma, \sigma'),$$

where in the last equality the operator acts on the first coordinate and Δ_d^- is the lower Laplacian associated with the weight function w_\downarrow defined in Example 2.7(3). This in turn implies by a similar argument to the one in Lemma 3.2(2) that

$$\mathcal{E}_n^{\downarrow,p}(\sigma, \sigma') = \tilde{\mathcal{E}}_n^{\downarrow,p'}(\sigma, \sigma'),$$

where $p' = \frac{p}{(1-p)(M-2)+1}$.

It is now possible to generalize the results proved for ESBWR to its lower analogue by repeating the arguments in previous Sections.

A Appendix

Proof of Claim 2.4. For every $f \in \Omega_{L^2}^k$

$$\begin{aligned} \|\delta_k f\|^2 &= \sum_{\tau \in X^k} w(\tau) |\delta_k f(\tau)|^2 \leq \sum_{\tau \in X^k} w(\tau) \cdot \binom{k}{2} \sum_{i=0}^k |f(\tau \setminus \tau_i)|^2 \\ &= \sum_{\sigma \in X^{k-1}} \binom{k}{2} \cdot \left(\sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) |f(\sigma)|^2 \leq \binom{k}{2} \cdot \left(\sup_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) \|f\|^2, \end{aligned}$$

which shows that (2.2) implies that δ_k is bounded. As for the other direction for any $\sigma \in X_{\pm}^{k-1}$ the function $\mathbb{1}_{\sigma}$ satisfies $\left\| \frac{1}{\sqrt{w(\sigma)}} \mathbb{1}_{\sigma} \right\|^2 = 1$ and therefore

$$\|\delta_k \mathbb{1}_{\sigma}\|^2 = \sum_{\tau \in X^k} w(\tau) \left| \delta_k \left(\frac{1}{\sqrt{w(\sigma)}} \mathbb{1}_{\sigma}(\tau) \right) \right|^2 = \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau).$$

Thus, whenever $\sup_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) = \infty$ the operator δ_k is not bounded. \square

Proof of Claim 2.5. If $\deg(\sigma) < \infty$ for every σ , then $\partial_k g(\sigma)$ is a finite sum for every $\sigma \in X_{\pm}^{d-1}$ and is thus well defined. If 2.2 holds, then for every $g \in \Omega_{L^2}^k$

$$\begin{aligned} \|\partial_k g\|^2 &= \sum_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \left| \sum_{v \triangleleft \sigma} w(v\sigma) g(v\sigma) \right|^2 \leq \sum_{\sigma \in X^{k-1}} \frac{1}{w(\sigma)} \left(\sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) \left(\sum_{v \triangleleft \sigma} w(v\sigma) |g(v\sigma)|^2 \right) \\ &\leq \sup_{\sigma \in X^{k-1}} \left(\frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) \cdot \left(\sum_{\sigma \in X^{k-1}} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) |g(\tau)|^2 \right) \\ &= k \cdot \sup_{\sigma \in X^{k-1}} \left(\frac{1}{w(\sigma)} \sum_{\tau \in \text{cf}(\sigma)} w(\tau) \right) \cdot \|g\|^2. \end{aligned}$$

\square

Proof of Claim 5.2. The equivalence of the first three conditions follows from:

$$\left\langle \Delta_{X \setminus A}^+ f, f \right\rangle_{X \setminus A} = \left\langle \delta_d^{X \setminus A} f, \delta_d^{X \setminus A} f \right\rangle = \left\langle \delta_d \tilde{f}, \delta_d \tilde{f} \right\rangle = \sum_{\tau \in X^d} \left(\sum_{i=0}^d f(\tau \setminus \tau_i) \cdot \chi_{\tau \setminus \tau_i \notin A_{\pm}} \right)^2, \quad (\text{A.1})$$

where $\langle \cdot, \cdot \rangle_{X \setminus A}$ is the inner product $\langle \cdot, \cdot \rangle$ restricted to $X^{d-1} \setminus A$. As for the last claim, the equivalence between $\ker \delta_d^{X \setminus A} = 0$ and $H_d(X, A) = 0$ follows directly from the definition of the relative homology, see [Hat02, Section 2.1]. \square

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